Lecture 2 (Section 1.2)

Row Echelon Form

Reduced Row Echelon Form

Elementary Row Operations

Row Reduction Algorithm

Solutions of Linear Systems

Existence and Uniqueness Questions

(Row) Echelon Form

- Nonzero row - row containing at least one nonzero entry
- Leading entry - leftmost nonzero entry (in a nonzero row)

Matrix in echelon form (or echelon matrix):

- All nonzero rows are above all zero rows
- Each leading entry is to the right of the leading entry of the above row
- All entries in a column below a leading entry are zero

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 5 & 0 & 7 & 6 \\ 0 & 0 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Reduced (Row) Echelon Form

Matrix in reduced echelon form (or reduced echelon matrix):

- It is in echelon form
- Each leading entry is 1
- Each leading entry is the only nonzero entry in its column

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary row operations

- Interchange - Interchange two rows
- Scaling - Multiply all entries in a row by a nonzero constant
- Replacement - Replace one row by the sum of itself and a multiple of another row

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Perform elementary row operations on A and get U, i.e., let U be row equivalent to A

- If U is in echelon form, we say U is an echelon form of A
- If U is in reduced echelon form, we say U is the reduced echelon form of A

Theorem: For any matrix, there is one and only one reduced echelon form.

Row reduction algorithm

Forward phase: echelon form

- Determine the leftmost nonzero column (pivot column)
- Select a nonzero entry in the pivot column (pivot)
- Interchange row to move this entry to the top position (pivot position)
- Use row replacement operations to create zeros in all positions below the pivot
- Ignoring (or covering) the row containing the pivot position, repeat the process until there are only zeros

Backward phase: reduced echelon form

- Use scaling operations to make pivots equal to 1
- Beginning with the rightmost pivot and working upward to the left, create zeros above each pivot

Invariance of pivot positions

- Backward phase does not change pivot positions
- The reduced echelon form is unique

$\Rightarrow$ the leading entries are always in the same positions in any echelon form of a given matrix

Linear Systems

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left\{\begin{array}{ccccl} x_1 & + & 2x_2 & & = 4 \\ & & & x_3 & = 5 \\ & & & 0 & = 0 \end{array}\right.$$

- pivot columns - basic variables: x_1, x_3
- the rest - free variables: x_2

$$\left\{\begin{array}{l} x_1 = 4 - 2x_2 \\ x_2 \text{ is free} \\ x_3 = 5 \end{array}\right.$$

$(4 - 2x_2, x_2, 5)$ is a solution for any real number $x_2 \in \mathbb{R}$

In a slightly different notation: $(4 - 2t, t, 5)$ for any real number $t \in \mathbb{R}$

Existence and Uniqueness

If the rightmost column is a pivot column, like in

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{cases} x_1 + 2x_2 & = 0 \\ x_3 & = 0 \\ 0 & = 1 \end{cases}$$

then the system is inconsistent.
If the rightmost column is not a pivot column, like in

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 + 2x_2 & = 4 \\ x_3 & = 5 \\ 0 & = 0 \end{cases}$$

or

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{cases} x_1 & = 4 \\ x_2 & = 5 \\ x_3 & = 0 \end{cases}$$

then the system is consistent.
If there are no free variables, then the system has a unique solution.

It is sufficient to have an echelon form to answer these questions. If a system is determined to be consistent, one can go ahead and find the reduced echelon form to solve the system (in other words, to find the solution set).