MATH 20F WINTER 2007 PRACTICE FINAL

MARCH 21

PROBLEM 1: Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvalues of A are 7 and -2.

a). Determine if A can be diagonalized. If it can be diagonalized, find a diagonalization of A, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$

b). Orthogonally diagonalize A, that is, find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^{-1}$.

c). Solve the equation $A^k\mathbf{x} = \mathbf{b}$, where k is a given integer.

SOLUTION:

1a). Following Example 3 of Section 7.1, we have

$$P = \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Note that your answer may be different than this depending on how you choose the free variables and how you arrange the columns of P.

1b). Following the example, we have

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix}, \qquad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Note again that the particular answer you obtained can be different.

1c). Since $A = UDU^{-1}$, we have $A^k = UD^kU^{-1}$. Taking inverse from the both sides, we have $(A^k)^{-1} = U(D^k)^{-1}U^{-1}$, or $\mathbf{x} = (A^k)^{-1}\mathbf{b} = U(D^k)^{-1}U^{-1}\mathbf{b}$. Finding the inverse of an orthogonal matrix is easy: $U^{-1} = U^T$, so is finding the inverse of power of a diagonal matrix:

$$D^{-k} = (D^k)^{-1} = \begin{bmatrix} \frac{1}{7^k} & 0 & 0 \\ 0 & \frac{1}{7^k} & 0 \\ 0 & 0 & \frac{1}{(-2)^k} \end{bmatrix}.$$

We conclude

$$\mathbf{x} = UD^{-k}U^T\mathbf{b} = \begin{bmatrix} 7^{-k} \\ 0 \\ 7^{-k} \end{bmatrix}.$$

PROBLEM 2: Let the following vectors be given:

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

a). Find an orthogonal basis for $H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
b). Find a basis for the orthogonal complement $H^\perp$ of H.
c). Find vectors $\mathbf{y} \in H$ and $\mathbf{z} \in H^\perp$ such that $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

SOLUTION:

2a). The orthogonal projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$ is

$$\hat{\mathbf{v}}_2 = \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_2 \cdot \mathbf{v}_1}\mathbf{v}_1 = -\frac{44}{17}\mathbf{v}_1.$$

Then the vector $\mathbf{u}_2 = \mathbf{v}_1 - \hat{\mathbf{v}}_2$ should be orthogonal to $\mathbf{v}_1$ and still in H.

$$\mathbf{u}_2 = \mathbf{v}_1 - \hat{\mathbf{v}}_2 = (1 + \frac{44}{17})\mathbf{v}_1 = \frac{61}{17}\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}.$$

Now the set $\{\mathbf{v}_1, \mathbf{u}_2\}$ or if you prefer, $\{\mathbf{v}_1, 17\mathbf{u}_2\}$ is an orthogonal basis for H.

2b). With the matrix $V = [\mathbf{v}_1\ \mathbf{v}_2]$, we have $H = \operatorname{Col} V$. So using the fundamental theorem, we have $H^\perp = (\operatorname{Col} V)^\perp = \operatorname{Nul} V^T$. A direct calculation gives

$$\mathbf{v}_3 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

is a basis for $\operatorname{Nul} V^T = H^\perp$.

2c). From the problem statement we see that $\mathbf{y} \in H$ is the orthogonal projection of $\mathbf{x}$ onto H and $\mathbf{z} \in H^\perp$ is the orthogonal projection of $\mathbf{x}$ onto $H^\perp$. There are at least three ways to calculate $\mathbf{y}$ and $\mathbf{z}$.

(i) The vectors $\mathbf{v}_1$, $\mathbf{u}_2$, and $\mathbf{v}_3$ together constitute an orthogonal basis for $\mathbb{R}^3$. We can expand $\mathbf{x}$ in terms of this basis as $\mathbf{x} = \alpha\mathbf{v}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{v}_3$. Then the vectors $\mathbf{y} = \alpha\mathbf{v}_1 + \beta\mathbf{u}_2$ and $\mathbf{z} = \gamma\mathbf{v}_3$ satisfy the conditions of the problem (see Theorem 5 of Section 6.2 and Example 1 of Section 6.3).

(ii) Since we have an orthogonal basis for H, we can calculate $\mathbf{y}$ by Theorem 8 of Section 6.3, and find $\mathbf{z}$ by $\mathbf{z} = \mathbf{x} - \mathbf{y}$.

(iii) The quickest method: $\{\mathbf{v}_3\}$ is trivially an orthogonal basis for $H^\perp$, since $H^\perp$ is one dimensional. So we can calculate $\mathbf{z}$ by Theorem 8 of Section 6.3, and find $\mathbf{y}$ by $\mathbf{y} = \mathbf{x} - \mathbf{z}$.

PROBLEM 3: Let

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

a). Find the area of the triangle whose vertices are $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.
b). If A is an orthogonal matrix, find the area of the triangle whose vertices are $A\mathbf{u}_1$, $A\mathbf{u}_2$, and $A\mathbf{u}_3$.

SOLUTION:

3a). Moving $\mathbf{u}_1$ to the origin, the area of the triangle is equal to

$$S = \frac{1}{2}|\det U| \qquad \text{with } U = [\mathbf{u}_2 - \mathbf{u}_1 \ \mathbf{u}_3 - \mathbf{u}_1].$$

We calculate

$$\det U = \begin{vmatrix} 4 & 2 \\ 1 & -3 \end{vmatrix} = -14,$$

so $S = |-14|/2 = 7$.

3b). Analogously to the above, we would have to calculate the determinant of

$$U' = [A\mathbf{u}_2 - A\mathbf{u}_1 \ A\mathbf{u}_3 - A\mathbf{u}_1] = [A(\mathbf{u}_2 - \mathbf{u}_1) \ A(\mathbf{u}_3 - \mathbf{u}_1)] = AU.$$

We have $\det(AU) = (\det A)(\det U)$, and since A is orthogonal,

$$1 = \det I = \det(A^T A) = (\det A^T)(\det A) = (\det A)^2.$$

So the area of the modified triangle is

$$S' = \frac{1}{2}|\det U'| = \frac{1}{2}|\det(AU)| = \frac{1}{2}|(\det A)(\det U)|$$
$$= \frac{1}{2}|\det A||\det U| = \frac{1}{2}|\det U| = S = 7,$$

where we used $|\det A| = 1$.

PROBLEM 4: Let A be a matrix such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^n$. Prove that A is an orthogonal matrix.

SOLUTION:

Using the linearity and the symmetricity of the inner product, we have

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} + 2\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\mathbf{x}\cdot\mathbf{y}.$$

and similarly,

$$\|A\mathbf{x}+A\mathbf{y}\|^2 = \|A\mathbf{x}\|^2 + \|A\mathbf{y}\|^2 + 2(A\mathbf{x})\cdot(A\mathbf{y}).$$

The condition $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^n$, implies that

$$(A\mathbf{x})\cdot(A\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}, \tag{1}$$

where we have used the above two equalities in combination with

$$\|A\mathbf{x}+A\mathbf{y}\|^2 = \|A(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}+\mathbf{y}\|^2, \qquad \text{and} \qquad \|A\mathbf{y}\| = \|\mathbf{y}\|.$$

Using $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^T\mathbf{y}$, the equation (1) can be written as $(A\mathbf{x})^T A\mathbf{y} = \mathbf{x}^T\mathbf{y}$, and since $(A\mathbf{x})^T = \mathbf{x}^T A^T$, we have

$$\mathbf{x}^T A^T A\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x}^T I\mathbf{y}.$$

where I is the identity matrix. This equality is true for *any* $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, so $A^T A$ should be equal to I, showing that A is orthogonal.

The above argument can be made more rigorous by taking $\mathbf{x} = \mathbf{e}_i$ and $\mathbf{y} = \mathbf{e}_k$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{R}^n$. One can show that for any $n \times n$ matrix B with elements b_{ik}, $\mathbf{e}_i^T B \mathbf{e}_k = b_{ik}$.

PROBLEM 5: Derive a formula for the least-squares solution of $A\mathbf{x} = \mathbf{b}$ when the columns of A are orthonormal.

SOLUTION:
The least squares problem is equivalent to the normal equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. Since A is orthogonal, we have $A^T A = I$, and so $\hat{\mathbf{x}} = A^T\mathbf{b}$.

PROBLEM 6: Mark each statement TRUE or FALSE. Briefly justify each answer.

a). An eigenvector of A corresponding to the eigenvalue α is a solution of the equation $(A - \alpha I)\mathbf{x} = \mathbf{0}$.
b). Similar matrices have the same eigenvalues.
c). An $n \times n$ matrix A is diagonalizable if A has n distinct eigenvalues.
d). An $n \times n$ matrix A is diagonalizable if and only if A has n distinct eigenvalues.
e). Any solution of $A^T A\mathbf{x} = A^T\mathbf{b}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$ only if A has linearly independent columns.

SOLUTION:

6a). TRUE. $A\mathbf{x} = \alpha\mathbf{x} \Leftrightarrow (A - \alpha I)\mathbf{x} = \mathbf{0}$
6b). TRUE. They have the same characteristic polynomials.
6c). TRUE. Distinct eigenvalues have linearly independent eigenvectors.
6d). FALSE. Look at the matrix A in Problem 1. It is a 3×3 matrix having 2 distinct eigenvalues but is diagonalizable.
6e). FALSE. Any solution of $A^T A\mathbf{x} = A^T\mathbf{b}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$ regardless of whether A has linearly independent columns.