MATH 20F WINTER 2007 PRACTICE FINAL

MARCH 21

GUIDELINES:

- *Please put your* **name**, **ID number**, *and* **TA's name** *on your blue book or exam sheet.*
- *No books, notes, or calculators are allowed.*
- *Write your solutions clearly and give explanations for your work. Answers without justifications will not be given credit.*
- *If any question is not clear, ask for clarification.*

PROBLEMS: (*Note*: The actual exam will have 7-8 problems.)

1. [*15pts*] Let the matrix A and the vector $\mathbf{b}$ be given by
$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$
The eigenvalues of A are 7 and -2.
   a). [*5pts*] Determine if A can be diagonalized. If it can be diagonalized, find a diagonalization of A, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$
   b). [*5pts*] Orthogonally diagonalize A, that is, find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^{-1}$.
   c). [*5pts*] Solve the equation $A^k\mathbf{x} = \mathbf{b}$, where k is a given integer.

2. [*15pts*] Let the following vectors be given:
$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$
   a). [*5pts*] Find an orthogonal basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
   b). [*5pts*] Find a basis for the orthogonal complement $H^\perp$ of H.
   c). [*5pts*] Find vectors $\mathbf{y} \in H$ and $\mathbf{z} \in H^\perp$ such that $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

(SEE OTHER SIDE)

3. [*10pts*] Let
$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$
   a). [*5pts*] Find the area of the triangle whose vertices are $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.
   b). [*5pts*] If A is an orthogonal matrix, find the area of the triangle whose vertices are $A\mathbf{u}_1$, $A\mathbf{u}_2$, and $A\mathbf{u}_3$.

4. [*8pts*] Let A be a matrix such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^n$. Prove that A is an orthogonal matrix. (*Hint*: Expand $\|\mathbf{x}+\mathbf{y}\|^2$ and $\|A\mathbf{x}+A\mathbf{y}\|^2$ to show that $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then write the inner products as matrix products and see the matrix in between $\mathbf{x}^T$ and $\mathbf{y}$ is equal to the identity since $\mathbf{x}$ and $\mathbf{y}$ are arbitrary.)

5. [*7pts*] Derive a formula for the least-squares solution of $A\mathbf{x} = \mathbf{b}$ when the columns of A are orthonormal.

6. [*10pts*] Mark each statement TRUE or FALSE. Briefly justify each answer.
   a). An eigenvector of A corresponding to the eigenvalue α is a solution of the equation $(A - \alpha I)\mathbf{x} = \mathbf{0}$.
   b). Similar matrices have the same eigenvalues.
   c). An $n \times n$ matrix A is diagonalizable if A has n distinct eigenvalues.
   d). An $n \times n$ matrix A is diagonalizable if and only if A has n distinct eigenvalues.
   e). Any solution of $A^TA\mathbf{x} = A^T\mathbf{b}$ is a least-squares solution of $A\mathbf{x} = \mathbf{b}$ only if A has linearly independent columns.