MATH 20F WINTER 2007 MIDTERM EXAM I

JANUARY 31

PROBLEM 1: Consider the following system of linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 &= 12 \\ x_2 + x_3 &= 5 \\ 3x_1 - 2x_2 + x_3 &= 11. \end{cases}$$

a). Write down the augmented matrix of the system, and use the row reduction algorithm to find a row echelon form of the matrix. Circle the pivot positions in the final matrix and list the pivot columns. Determine whether the system is consistent. If the system is consistent, find all solutions of the system. Write it in parametric vector form.

b). Write the system in the form $A\mathbf{x} = \mathbf{b}$, with A being the coefficient matrix of the system. Determine if there exist $\mathbf{c}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{c}$ has *no* solution, and if there is such $\mathbf{c}$, give an example. Describe the set of all $\mathbf{c}$ in $\mathbb{R}^3$ for which $A\mathbf{x} = \mathbf{c}$ *does* have a solution.

SOLUTION:

1a). The augmented matrix is

$$\bar{A} = \begin{bmatrix} 1 & 1 & 2 & 12 \\ 0 & 1 & 1 & 5 \\ 3 & -2 & 1 & 11 \end{bmatrix},$$

and *a row echelon form* can be found by the following sequence of elementary row operations:

$$\begin{bmatrix} 1 & 1 & 2 & 12 \\ 0 & 1 & 1 & 5 \\ 3 & -2 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 12 \\ 0 & 1 & 1 & 5 \\ 0 & -5 & -5 & -25 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 1 & 2 & 12 \\ 0 & \mathbf{1} & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, the pivot positions are indicated by the boldface numbers. The pivot columns are the columns 1 and 2. (Note that since echelon form is not unique, the particular echelon form you obtained can be different than this, but the pivot positions should be the same.)

Since the last column is not a pivot column, the system is *consistent*. We can perform one more elementary row operation to obtain the reduced row echelon form of the augmented matrix:

$$\begin{bmatrix} \mathbf{1} & 1 & 2 & 12 \\ 0 & \mathbf{1} & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 0 & 1 & 7 \\ 0 & \mathbf{1} & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the *solution* is:

$$\begin{cases} x_1 = 7 - x_3 \\ x_2 = 5 - x_3 \\ x_3 \text{ is free,} \end{cases}$$

or in *parametric vector form*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 - x_3 \\ 5 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

1b). The coefficient matrix A and the right hand side vector $\mathbf{b}$ are given by

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 3 & -2 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 12 \\ 5 \\ 11 \end{bmatrix},$$

so that the above system can be written as $A\mathbf{x} = \mathbf{b}$.

Let us investigate for what values of $\mathbf{c}$ in $\mathbb{R}^3$ the system $A\mathbf{x} = \mathbf{c}$ is consistent. The sequence of elementary row operations remains the same as in a), we just have to focus on the last column:

$$\begin{bmatrix} 1 & 1 & 2 & c_1 \\ 0 & 1 & 1 & c_2 \\ 3 & -2 & 1 & c_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & c_1 \\ 0 & 1 & 1 & c_2 \\ 0 & -5 & -5 & c_3 - 3c_1 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 1 & 2 & c_1 \\ 0 & \mathbf{1} & 1 & c_2 \\ 0 & 0 & 0 & c_3 - 3c_1 + 5c_2 \end{bmatrix}.$$

Now we see that if $c_3 - 3c_1 + 5c_2 \neq 0$, the system $A\mathbf{x} = \mathbf{c}$ is inconsistent. For example, $c_1 = c_2 = c_3 = 1$ would give *inconsistent* system.

That $A\mathbf{x} = \mathbf{c}$ does have a solution is equivalent to saying that $\mathbf{c}$ can be written as

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ 3c_1 - 5c_2 \end{bmatrix}.$$

PROBLEM 2: Consider the matrix A and vector $\mathbf{b}$ given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 562.23 \\ 362.82 \end{bmatrix}.$$

a). Are the columns of A linearly independent or linearly dependent? Justify your answer. Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

b). Is $\mathbf{b}$ in the set spanned by the columns of A? Do the columns of A span $\mathbb{R}^2$?

SOLUTION:

2a). The columns of A are linearly *dependent* because A has more columns than its rows. Another way of saying it is that A has at least one non-pivot column, so the homogeneous system $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions, which is equivalent to the columns of A being linearly dependent.

In order to solve the homogeneous equation, let us perform the row reduction algorithm to find the reduced echelon form of A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 1 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 2 & 3 \\ 0 & \mathbf{2} & -5 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 0 & 2 \\ 0 & \mathbf{2} & -5 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & 0 & 2 \\ 0 & \mathbf{1} & -5/2 \end{bmatrix}.$$

If we choose x_3 as a free variable, all solutions to $A\mathbf{x} = \mathbf{0}$ can be described as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 2.5x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 2.5 \\ 1 \end{bmatrix}.$$

2b). Since A has a pivot in every row, the columns of A *span* $\mathbb{R}^2$. The vector $\mathbf{b} = \begin{bmatrix} 562.23 \\ 362.82 \end{bmatrix}$ is certainly in $\mathbb{R}^2$, so $\mathbf{b}$ is *in* the set spanned by the columns of A.

PROBLEM 3: Let $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear mappings given by

$$S(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 \\ -x_2 \end{bmatrix}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and

$$T(\mathbf{x}) = \begin{bmatrix} 7x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

a). Is S onto? Is it one-to-one? Justify your answer.
b). Find the standard matrix of the composition $L = S \circ T$, that is, the mapping $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $L(\mathbf{x}) = S(T(\mathbf{x}))$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

SOLUTION:

3a). We have to check whether every vector $\mathbf{b}$ in $\mathbb{R}^2$ can be written as $\mathbf{b} = S(\mathbf{x})$ for some $\mathbf{x}$ in $\mathbb{R}^2$, and since S is linear, this question is equivalent to asking whether the columns of the (standard) matrix of S span $\mathbb{R}^2$. The columns of this matrix is equal to the images of $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have

$$S(\mathbf{e_1}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad S(\mathbf{e_2}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

so the matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Since A has a pivot in every row, S is *onto*.

Similarly, the question of whether S is one-to-one is equivalent to the question of whether the columns of A are linearly independent. Since every column of A is a pivot column, S is *one-to-one*.

3b). We demonstrate two ways to find the standard matrix of the composition $L = S \circ T$.

(i) *Direct calculation.* Let us find the image of $\mathbf{e}_1$ under the mapping L:

$$L(\mathbf{e}_1) = S(T(\mathbf{e}_1)) = S(\begin{bmatrix} 7 \cdot 1 - 2 \cdot 0 \\ -2 \cdot 1 + 5 \cdot 0 \end{bmatrix}) = S(\begin{bmatrix} 7 \\ -2 \end{bmatrix}) = \begin{bmatrix} 7 + 2 \cdot (-2) \\ -(-2) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Similarly, we have

$$L(\mathbf{e}_2) = S(T(\mathbf{e}_2)) = S(\begin{bmatrix} -2 \\ 5 \end{bmatrix}) = \begin{bmatrix} 8 \\ -5 \end{bmatrix}.$$

So the standard matrix of L is

$$C = \begin{bmatrix} 3 & 8 \\ 2 & -5 \end{bmatrix}.$$

(ii) *Matrix multiplication.* We know from a) that the standard matrix of S is A. If we find the standard matrix (call it B) of T, then the standard matrix of the composition $L = S \circ T$ is $C = AB$. Since

$$T(\mathbf{e}_1) = \begin{bmatrix} 7 \\ -2 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{e}_2) = \begin{bmatrix} -2 \\ 5 \end{bmatrix},$$

the standard matrix of T is

$$B = \begin{bmatrix} 7 & -2 \\ -2 & 5 \end{bmatrix}.$$

Now we use matrix multiplication to find C:

$$C = AB = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 7 & -2 \\ -2 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \cdot 7 + 2 \cdot (-2) & 1 \cdot (-2) + 2 \cdot 5 \\ 0 \cdot 7 + (-1) \cdot (-2) & 0 \cdot (-2) + (-1) \cdot 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 2 & -5 \end{bmatrix}.$$

PROBLEM 4: Mark each statement TRUE or FALSE. Justify each answer.

a). When $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, Span $\{\mathbf{u}, \mathbf{v}\}$ is a plane.

b). The columns of a $k \times n$ matrix are linearly independent if and only if it has n pivot columns.

c). If A is a $k \times n$ matrix, then the range of $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^k$.

d). The columns of the standard matrix for a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ are the images of the columns of the $n \times n$ identity matrix.

e). The linear mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is onto if and only if its standard matrix has n pivot columns.

SOLUTION:

4a). FALSE. When $\mathbf{u}$ and $\mathbf{v}$ are on the same line (in other words, $\mathbf{u} = \alpha\mathbf{v}$ for some scalar α), Span$\{\mathbf{u}, \mathbf{v}\}$ is a line.

4b). TRUE. There are *no* free variables, so the homogeneous equation will have *only* the trivial solution.

4c). FALSE. The codomain is $\mathbb{R}^k$ but in general the range is *not* equal to $\mathbb{R}^k$. The range of $\mathbf{x} \mapsto A\mathbf{x}$ is the set spanned by the columns of A. So if there is a row in A that has no pivot, one can construct $\mathbf{b}$ in $\mathbb{R}^k$ such that $A\mathbf{x} = \mathbf{b}$ has *no* solution, meaning that $\mathbf{b}$ is not in the range of $\mathbf{x} \mapsto A\mathbf{x}$.

4d). TRUE. Using linearity of the mapping one can prove this fact. Observe that the columns of I_n is in $\mathbb{R}^n$, so at least it makes sense to speak of the images of the columns of I_n under this mapping.

4e). FALSE. The standard matrix of the linear mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a $k \times n$ matrix. T is onto if and only if the columns of its standard matrix span $\mathbb{R}^k$, and the columns of the $k \times n$ matrix span $\mathbb{R}^k$ if and only if it has k pivot columns. But in general n and k can be different.