MATH 20F WINTER 2007 FINAL EXAM

MARCH 21

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GUIDELINES:

- *Please put your* **name**, **ID number**, *and* **TA's name** *on your blue book.*
- *No books, notes, or calculators are allowed.*
- *Write your solutions clearly and give explanations for your work. Answers without justifications will not be given credit.*
- *If any question is not clear, ask for clarification.*
- *Please draw the following table on the* **inner side** *of the* **front cover** *of your blue book.*

PROBLEMS:

1. [*15pts*]

   a). [*5pts*] Find the inverse of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

   b). [*5pts*] Find the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto the (one-dimensional) subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

   c). [*5pts*] Determine the rank of $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

2. [*10pts*] Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mappings given by

$$S(\mathbf{x}) = \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

   and

$$L(\mathbf{x}) = \begin{bmatrix} 5x_2 \\ -4x_1 + 9x_2 \end{bmatrix}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   a). [*5pts*] Is S onto? Is it one-to-one? Justify your answer.

   b). [*5pts*] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping such that $L = S \circ T$, that is, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping such that $L(\mathbf{x}) = S(T(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^2$. Find the standard matrix of T.

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3. [*10pts*] Let the matrix A and the vectors $\mathbf{x} \in \mathbb{R}^4$ and $\mathbf{z} \in \mathbb{R}^3$ be given by

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 5 & -8 \\ -1 & -4 & 7 \\ 3 & 1 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} 122 \\ -244 \\ -122 \end{bmatrix}.$$

a). [*5pts*] Find a *basis* for and the *dimension* of $\operatorname{Col} A$. Is $\mathbf{x}$ in $\operatorname{Col} A$?
b). [*5pts*] Find a *basis* for and the *dimension* of $\operatorname{Nul} A$. Is $\mathbf{z}$ in $\operatorname{Nul} A$?

4. [*15pts*] Let the following vectors be given:

$$\mathbf{u}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

It is easy to show that both $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ are bases for $\mathbb{R}^2$.
a). [*5pts*] Find the change-of-coordinate matrix from $\mathcal{U}$ to $\mathcal{V}$, that is, find the matrix P such that $[\mathbf{x}]_\mathcal{V} = P[\mathbf{x}]_\mathcal{U}$ for all $\mathbf{x} \in \mathbb{R}^2$.
b). [*5pts*] If the coordinate vector of $\mathbf{x} \in \mathbb{R}^2$ relative to the basis $\mathcal{U}$ is given by

$$[\mathbf{x}]_\mathcal{U} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

find $[\mathbf{x}]_\mathcal{V}$, that is, the coordinate vector of $\mathbf{x}$ relative to the basis $\mathcal{V}$.
c). [*5pts*] Find the change-of-coordinate matrix from $\mathcal{V}$ to $\mathcal{U}$, that is, find the matrix Q such that $[\mathbf{x}]_\mathcal{U} = Q[\mathbf{x}]_\mathcal{V}$ for all $\mathbf{x} \in \mathbb{R}^2$.

5. [*20pts*] Let the matrices A, B, and the vector $\mathbf{x}$ be given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}.$$

a). [*5pts*] Find all eigenvalues and eigenvectors of A. Is A diagonalizable?
b). [*5pts*] Find all eigenvalues and eigenvectors of B.
c). [*5pts*] Orthogonally diagonalize B.
d). [*5pts*] Suppose that H and G are the eigenspaces of B. Then find $\mathbf{y} \in H$ and $\mathbf{z} \in G$ such that $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

6. [*10pts*] Mark each statement TRUE or FALSE. Briefly justify each answer.
a). A number α is an eigenvalue of A if and only if the equation $(A - \alpha I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
b). $\det(A + B) = \det A + \det B$
c). The dimension of an eigenspace of a matrix equals the multiplicity of the corresponding eigenvalue.
d). Eigenvectors corresponding to different eigenvalues of a symmetric matrix are orthogonal.
e). The general least-squares problem is to find an $\mathbf{x}$ that makes $A\mathbf{x}$ as close as possible to $\mathbf{b}$.

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7. [*10pts*] Let A and B be $n \times n$ matrices.
   a). [*5pts*] Use determinants to get a proof of the fact that if $AB = I$, then both A and B are invertible.
   b). [*5pts*] Show that if $A \neq 0$, $B \neq 0$, and $AB = 0$, then $\det A = \det B = 0$. (*Hint*: What are the columns of AB? What does the condition $AB = 0$ say about the columns of B? First consider the possibility $\det B \neq 0$ and use the fact that $A = 0$ if $\operatorname{Nul} A = \mathbb{R}^n$. Now consider the condition $B^T A^T = 0$ and the possibility $\det A^T \neq 0$.)

8. [*10pts*] Let H be a subspace of $\mathbb{R}^n$, and let A be a matrix such that $\|A\mathbf{y}\| = \|\mathbf{y}\|$ for any $\mathbf{y} \in H$, and $A\mathbf{z} = \mathbf{0}$ for any $\mathbf{z} \in H^\perp$. Prove that for any $\mathbf{x} \in \mathbb{R}^n$, the vector $A^T A\mathbf{x}$ is equal to the orthogonal projection of $\mathbf{x}$ onto H, as follows:
   a). [*5pts*] Any vector $\mathbf{x} \in \mathbb{R}^n$ can be written as $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$ with $\hat{\mathbf{x}} \in H$ being the orthogonal projection of $\mathbf{x}$ onto H, and $\mathbf{z} \in H^\perp$. Then it is clear that $\mathbf{x} \in H^\perp$ if and only if $\hat{\mathbf{x}} = 0$. Use this observation to show that $\operatorname{Nul} A = H^\perp$. Explain why this implies $\operatorname{Col} A^T = H$. Show that $A^T A\mathbf{x} = A^T A\hat{\mathbf{x}}$.
   b). [*5pts*] Show that $(A\mathbf{u}) \cdot (A\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ for any $\mathbf{u}, \mathbf{v} \in H$. Further manipulate the equality to show that $\mathbf{u}^T(A^T A\mathbf{v} - \mathbf{v}) = 0$ for any $\mathbf{u}, \mathbf{v} \in H$. Is $A^T A\mathbf{v} - \mathbf{v}$ in $H^\perp$? Using a result from a), show that $A^T A\mathbf{v} - \mathbf{v} \in H$. What do these two conditions imply? From a) we have $A^T A\mathbf{x} = A^T A\hat{\mathbf{x}}$ where $\hat{\mathbf{x}} \in H$ is the orthogonal projection of $\mathbf{x}$ onto H. What can you say about $A^T A\hat{\mathbf{x}}$ now?