1. (a) Two points \( P = (x, y) \) and \( Q = (x', y') \) are chosen at random inside a square having sides of length \( s \). What is the expected value of \( D^2 \), the square of the distance between \( P \) and \( Q \)?

(b) Consolidated Industries is marketing a new product and wish to decide how many to manufacture. They estimate that each item sold will return a profit of \( m \) dollars; each one not sold represents a loss of \( n \) dollars. They also suspect that the demand for the product, \( V \), will have the density

\[
f(v) = \lambda e^{-\lambda v}, \ v > 0, \ \lambda > 0.
\]

How many items should the company produce if they want to maximize their expected profit (\( \lambda, m, n \) are assumed known)?

2. Let \( X_1, X_2, \ldots, X_n \) have a joint multinomial distribution

\[
f(x_1, x_2, \ldots, x_m | p_1, p_2, \ldots, p_m) = \frac{n!}{\prod_{i=1}^{m} x_i!} \cdot \prod_{i=1}^{m} p_i^{x_i}.
\]

(a) Find the maximum likelihood estimators of \( p_1, p_2, \ldots, p_m \).

(b) Derive the asymptotic sampling distribution of

\[
\frac{\hat{p}_j - p_j}{\sqrt{\frac{\hat{p}_j (1 - \hat{p}_j)}{n}}}, \text{ for some fixed } j = j_0, \text{ say.}
\]

(c) Before carrying out an experiment, it is desired to evaluate the sample size, \( n \), such that any 95 percent confidence interval for \( p_{j_0} \) constructed from the observed data will have width no greater than .06. Find the \( n \) that guarantees this.

3. A key ring has \( N \) keys. One of them fits a lock, but it is unknown which.

(a) If the keys are tried one-by-one with replacement, find the expected number of keys to be tried to find the key that fits the lock.
3. (b) If the keys are tried without replacement, find the expected number of keys to be tried to find the key that fits the lock.

(c) The number of particles entering a counter in a certain time period has a Poisson distribution with parameter \( \lambda \). A voltage is produced by multiplying the number of entering particles by a non-negative factor chosen independently of the entering number of particles, and with distribution having the density

\[
f(x) = \frac{1}{(1 + x)^2}, \quad x \geq 0
\]

\[
= 0, \text{ elsewhere.}
\]

Find the probability that the resulting voltage is less than one.

4. Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed exponential random variables with parameter \( \lambda \).

(a) Show that \( T = \sum_{i=1}^{n} X_i \) is a sufficient statistic for \( \lambda \).

(b) By quoting a general result on completeness, state why \( T \) is also complete.

(c) Show that \( S(X_1) = I_{\{X_1 \leq x\}} \) is an unbiased estimator of \( q(\lambda) = 1 - e^{-\lambda x} \), where \( I_{\{X_1 \leq x\}} \) is the usual indicator function of the set \( \{X_1 \leq x\} \).

(d) Assuming that \( \frac{X_1}{T} \) is independent of \( T \) and has a Beta(1, \( n - 1 \)) distribution, find the unique minimum variance unbiased estimator of \( q(\lambda) \).

(Note: the Beta(\( \alpha, \beta \)) density is given by \( \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} \) for \( 0 < x < 1 \) and 0 elsewhere.)

5. Suppose that \( (X_i, Y_i) \) \( i = 1, \ldots, n, \) is a sequence of bivariate normal random vectors with \( EX = \mu_x, \quad EY = \mu_y, \quad \text{Var}(X) = \text{Var}(Y) = 1 \) and \( \text{Cov}(X, Y) = \rho \), where \( -1 < \rho < 1 \).

(a) What is the likelihood ratio test statistic for testing \( H_0 : \mu_x = \mu_y \) vs. \( H_a : \mu_x \neq \mu_y \)?

(b) What is the distribution of the statistic in (a) under \( H_0 \)?
6. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from density function $f(x - \theta)$, where

$$f(x) = \begin{cases} 
1, & -\frac{1}{2} < x < \frac{1}{2} \\
0, & \text{otherwise.}
\end{cases}$$

(a) Show that $(X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for $\theta$ (where $X_{(1)}$ is the minimum of $X_1, \ldots, X_n$ while $X_{(n)}$ is the maximum).

(b) Show that $X_{(n)} - X_{(1)}$ is ancillary for $\theta$.

(c) Show that $\bar{X}_n$ (the sample mean) is unbiased for $\theta$.

(d) Can you come up with a better unbiased estimator than $\bar{X}_n$? Is your estimator an UMVUE? Why?

7. Two laboratories each take $n$ measurements on the same standard $\mu$. Consider the model

$$Y_{ij} = \mu + \varepsilon_{ij}, \quad i = 1, 2; \quad j = 1, \ldots, n,$$

where the $\varepsilon_{ij}$ are independent random variables with mean 0 such that $\text{Var}(\varepsilon_{1j}) = \sigma_1^2$, $\text{Var}(\varepsilon_{2j}) = \sigma_2^2$.

(a) Suppose it is known that $\frac{\sigma_2^2}{\sigma_1^2} = 4$. Show that the minimum variance linear unbiased estimator of $\mu$ is given by

$$\hat{\mu} = \frac{4\bar{Y}_1 + \bar{Y}_2}{5}, \quad \text{where} \quad \bar{Y}_i = \frac{1}{n} \sum_{j=1}^{n} Y_{ij}.$$

(b) Generalize the above result to the case where there are $k$ laboratories and the errors for the $i^{th}$ laboratory are all independent with mean 0 and variance $a_i \sigma^2 (i = 1, \ldots, k)$, where $a_i$ are known constants such that $\sum_{i=1}^{k} a_i = 1$. Discuss how you would estimate $\sigma^2$.

8. (a) A random sample of $n$ measurements $X_1, X_2, \ldots, X_n$ is drawn from a distribution with a continuous cumulative distribution function $F(x)$. Denote by $U_i (i = 1, 2, \ldots, n)$ the order statistics. Describe briefly a method of setting symmetrical $100(1 - \alpha)\%$ confidence limits for the median of the unknown $F(x)$.

(b) Ten samples of a test of milk were measured for strontium -90 concentration in micromicrocuries per litre. Neither the smallest nor the largest measurement was available, but the rest in order of size were 7.1, 8.2, 8.4, 9.1, 9.8, 9.9, 10.5, 11.3. Determine $P(U_3 < k_{\frac{1}{2}} < U_8)$ where $k_{\frac{1}{2}}$ is the median, $U_3 = 8.2$, and $U_8 = 10.5$. 

3