McGill University
Department of Mathematics and Statistics

Pure $\beta$

Date: Friday, September 13, 1996
Time: 13:00 - 17:00

Instructions: The seven best answers will count for your final grade.

1. Assume the well-known fact that every element of the symmetric group $S_n$ is a product of disjoint cycles. Show that two elements are conjugate if and only if they are the product of the same number of disjoint cycles of the same lengths. One way of doing this starts by showing that if $\sigma \in S_n$ is arbitrary and $\tau = (i_1 \ i_2 \ \cdots \ i_k)$ is a cycle, then
   \[
   \sigma \tau \sigma^{-1} = (\sigma i_1 \ \sigma i_2 \ \cdots \ \sigma i_k)
   \]

2. Prove that if $\mathcal{A}$ is a category, $T : \mathcal{A} \to \text{Set}$ a functor and $A$ an object of $\mathcal{A}$, then there is an equivalence between the set of natural transformations $\text{Hom}(A, -) \to T$ and elements of the set $T(A)$. Formulate (but do not prove) what it means for this equivalence to be natural in $A$.

3. This problem concerns the polynomial $f(x) = x^4 - 2$ over the field $\mathbb{Q}$ of rational numbers.
   (a) Show that $f$ is irreducible.
   (b) Show that the splitting field of $x^4 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}[i, \alpha]$ where $\alpha$ is a real 4th root of 2.
   (c) What is the degree $[\mathbb{Q}[i, \alpha] : \mathbb{Q}]$?
   (d) Show that $f$ is irreducible over $\mathbb{Q}[i]$.
   (e) Let $G$ be the galois group. Show that there is a $\sigma \in G$ such that $\sigma(i) = i$ and $\sigma(\alpha) = i\alpha$.
   (f) Show that there is a $\tau \in G$ with $\tau(i) = -i$ and $\tau(\alpha) = \alpha$.
   (g) Show that $\sigma^3 \leq \tau = \tau \leq \sigma$.
   (h) What group is $G$?
4. In this problem, \( T \) denotes the unit circumference, equipped with the usual Lebesque measure. Let \( f(e^{i\theta}) \in L_1(T) \), put \( a_n = \frac{1}{2\pi} \int_{-\pi}^\pi e^{-in\theta} f(e^{i\theta})d\theta \), and suppose that
\[
\sum_{-\infty}^{\infty} |a_n| < \infty.
\]
Show that there is a function \( g(e^{i\theta}) \in L_2(T) \) with
\[
f(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^\pi g(e^{i(\theta-t)})g(e^{it})dt \quad \text{a.e.,} \quad \theta \in T.
\]
(Hint: For each \( n \) take a \( b_n \) such that \( b_n^2 = a_n \) and note that \( \sum_{-\infty}^{\infty} |b_n|^2 < \infty \). Obtain \( g(e^{i\theta}) \) from the sequence \( \{b_n\} \), justifying your steps.)

5. Let \( \sigma \) be a finite regular signed Borel measure on \( I = [0, 1] \), and suppose that for each function \( \varphi \) continuous on \( I \) we have
\[
\left| \int_I \varphi d\sigma \right| \leq \int_I (e^{\varphi(t)} - 1) dt.
\]
Show that there is a function \( f \in L_1(I) \) with
\[
\int_I \varphi d\sigma = \int_I \varphi(t)f(t)dt
\]
for all such \( \varphi \).
(Hint: It suffices to show that \( \sigma(E) = 0 \) for all closed sets \( E \subseteq I \) of Lebesque measure 0 (why?). Show that, given such an \( E \) and \( \varepsilon > 0 \), we can get \( \mathcal{O} \supseteq E \) open in \( I \) with Lebesque measure of \( \mathcal{O} < \varepsilon \) and \( |\sigma|(\mathcal{O} \sim E) < \varepsilon \). By making a suitable choice of \( \varphi \), show that then \( |\sigma(E)| \leq \varepsilon \varepsilon \).

6. Let \( E \subseteq C \) be compact and have a connected complement in \( C \). Suppose that \( f(z) \), analytic for \( |z| < 1 \) and continuous for \( |z| \leq 1 \), has \( f(z) \in E \) for all \( z \) of modulus 1. Show that then \( f(z) \in E \) for all \( z \) in the unit disk.
(Hint: Suppose that \( f(z_0) = w_0 \notin E \) for some \( z_0 \), \( |z_0| < 1 \). Since \( C \sim E \) is open and connected, there is a broken line path \( \Lambda \) from \( w_0 \) out towards \( \infty \) which does not encounter \( E \). Let \( \rho > 0 \) be the shortest distance from points on \( \Lambda \) (sic!) to \( E \) and, with \( \gamma \) the circle of radius \( \rho/2 \) about \( w_0 \), put
\[
\Psi(w) = \frac{1}{2\pi i} \oint_{\gamma} \frac{3dw}{w-w} \quad , \quad w \notin \gamma.
\]
Check that \( \Psi(w) = 0 \) on \( E \) and that \( \Psi(w_0) = 3 \). Explain why there is a polynomial \( P(w) \) with \( |\Psi(w) - P(w)| \leq 1 \) for \( w \in E \) and for \( w = w_0 \). Look then at the function \( P(f(z)) \) for \( |z| = 1 \) and for \( z = z_0 \).
7. (a) On the unit circle
\[ S^1 = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 = 1\} \]
compute the integral of the 1-form
\[ \alpha = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx. \]

(b) Prove that \(d\alpha = 0\).

(c) Prove that there cannot exist a smooth function \(f : S^1 \rightarrow \mathbb{R}\) for which \(\alpha = df\).

(d) Prove that \(H^1(S^1, \mathbb{R}) \neq 0\).

8. Define \(V = \{v_1, v_2 \in \mathbb{R}^n \times \mathbb{R}^n : |v_1|^2 = 1, |v_2|^2 = 1, \langle v_1, v_2 \rangle = 0\}\).

(a) Prove that \(V\) is a smooth manifold of dimension \(2n - 3\).

(b) Prove also that \(V\) is compact.