INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.
[ALG. 1] (a) Prove that $G = (\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/3\mathbb{Z})$ is infinite.
(b) Show that there is a surjective group homomorphism $h : G \to S_3$.
(c) Show that any group homomorphism $h : G \to \mathbb{Z}/5\mathbb{Z}$ is trivial.

[ALG. 2] Let $F : \text{TopSp} \to \text{Set}$ be the forgetful functor from the category of topological spaces to the category of sets. Show that $F$ has both a right adjoint and a left adjoint.

[ALG. 3] Prove that a finite group $G$ is nilpotent if and only if every maximal proper subgroup is normal. Conclude that, if $G$ is a finite nilpotent group, then every proper normal subgroup has prime index.

[ALG. 4] Determine the Galois group of the polynomial $x^5 - x - 1$ over $\mathbb{Q}$. Also find the discriminant of this polynomial.
[AN. 1] Let \( f \) be a uniformly continuous function on \( \mathbb{R} \). Suppose that \( f \in L^p \) for some \( p, 1 \leq p < \infty \). Prove that \( f(x) \to 0 \) as \( |x| \to \infty \).

[AN. 2] (a) State the Radon-Nikodym Theorem.
(b) Let \((X_j, \mathcal{M}_j, \mu_j)\) and \((X_j, \mathcal{M}_j, \lambda_j)\) be \( \sigma \)-finite measure spaces for \( j = 1, 2 \), where the \( \mathcal{M}_j \)'s are \( \sigma \)-algebras. Suppose that each \( \mu_j \) is absolutely continuous relative to \( \lambda_j \). Prove that \( \mu_1 \otimes \mu_2 \) is absolutely continuous relative to \( \lambda_1 \otimes \lambda_2 \), and find the Radon-Nikodym derivative of \( \mu_1 \otimes \mu_2 \) relative to \( \lambda_1 \otimes \lambda_2 \).

[AN. 3] Let \( C^1[0, 1] \) be the space of continuously differentiable functions on \([0, 1]\). Define a norm on this space by
\[
||f||_1 = \max_{x \in [0, 1]} |f(x)| + \max_{x \in [0, 1]} |f'(x)|.
\]
(a) Verify that \( ||f||_1 \) is indeed a norm on \( C^1[0, 1] \).
(b) Let \( f_n(x) = \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \) for \( x \in [0, 1] \) and each natural number \( n \). Does \( (f_n) \) converge on \( C^0[0, 1] \) given the sup norm? Justify.
(c) Does \( (f_n) \) as just defined converge in \( C^1[0, 1] \) given the norm above? Justify.

[AN. 4] Let \( (f_n) \) be a sequence of analytic functions defined on the open unit disc \( D \) such that \( (f_n) \) converges to a function \( f \) and that the convergence is uniform on every compact subset of \( D \). Show that \( f \) is analytic.
[GT. 1] Construct an atlas of \( C^\infty \)-compatible charts on the 3-sphere \( S^3 \).

[GT. 2] Compute the de Rham cohomology of the 3-sphere \( S^3 \).
You may assume without proof the fact that

\[
H^k_{dR}(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}
\]

\[
H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & \text{otherwise} \end{cases}
\]

[GT. 3] Suppose that \( X \) is a complete regular space, \( A \) is a closed subset of \( X \), \( B \) is a compact subset of \( X \) and \( A \cap B = \emptyset \). Show that there is a continuous function \( f : X \to [0, 1] \) such that \( f(x) = 0 \) for all \( x \in A \) and \( f(x) = 1 \) for all \( x \in B \).

[GT. 4] Let \( S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \} \) and \( K = \{(x_1, x_2, x_3, x_4) \in S^3 : x_1 = x_2 = 0 \} \). Prove that

\[
\pi_1(S^3 \setminus K) = \pi_1(\mathbb{R}^3 \setminus K) = \mathbb{Z}.
\] (Here we identify \( S^3 \setminus \{N\} \) with \( \mathbb{R}^3 \) using the stereographic projection, with \( N \) being the north pole.)