As in the ALPHA paper, you are given a choice of questions to answer. However, in this paper (unlike in the ALPHA paper), additional work on questions over and above the required number will be taken into account for the overall grade.

**ALGEBRA**

Answer three of the four questions [1], [2], [3], and [4].

[1] Give a presentation of the ring $\mathbb{Z} \times \mathbb{Z}$: specify an ideal $(a, b, \ldots)$ of $\mathbb{Z}[x, y]$ by giving its generators $a, b, \cdots$ so that $\mathbb{Z} \times \mathbb{Z}$ is isomorphic to $\mathbb{Z}[x, y]/(a, b, \cdots)$. Give sufficient reasons for your answer.

[2] Let $R$ be a ring, $M$ a (left) $R$-module.

(a) Define: $M$ is noetherian.

(b) Prove or disprove: If $N \subset M$ is a submodule of $M$ such that $N$ and $M/N$ are noetherian, then $M$ is noetherian.

[3] Let $p$ and $q$ be any prime numbers, possibly equal to each other. Prove that no group of order $p^2 q$ is simple.

[4] (a) State the Fundamental Theorem of Galois Theory.

(b) State the basic facts concerning the Galois group over $\mathbb{Q}$ of the cyclotomic equation $x^p - 1 = 0$ for a prime number $p$. Take a look at the next part (c) and use it as a guide. Give details that are, or may be, relevant for the proof of the fact in (c).

(c) Let $p$ be a prime number, and let $\zeta$ be the complex number $\zeta = e^{2\pi i/p}$. Show that every subfield of the field $\mathbb{Q}(\zeta)$ is of the form $\mathbb{Q}(\theta)$ where $\theta$ is a quantity of the form

$$\theta = \sum_{k=0}^{\ell-1} \zeta^{(sk)}$$

with suitable positive integers $\ell$ and $s$. 

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ANALYSIS

Answer three of the four questions [5], [6], [7] and [8].

[5]  (a) State the Weierstrass approximation theorem.
  (b) Let \( \varepsilon > 0 \) and let \( f : [0, 1] \to \mathbb{R} \) have continuous
derivative \( f' \) on \([0, 1]\). Prove that there exists a polynomial \( p \) such that
\( |f(x) - p(x)| \leq \varepsilon \) and \( |f'(x) - p'(x)| \leq \varepsilon \)
for all \( x \in [0, 1] \). (Hint: use (a).)

[6]  Let the \( f_n \) and \( f \) be Lebesgue measurable and finite on \([0, 1]\), and suppose that
(i) \( \int_0^1 |f_n(x)|^2 \, dx \leq C < \infty \) for all \( n \);
(ii) \( \int_0^1 |f(x) - f_n(x)| \, dx \to 0 \).
  (a) Show that \( \int_0^1 |f(x)|^2 \, dx \leq C \). (Hint: Extract an appropriate
subsequence of \( \{ f_n \} \).)
  (b) Show that \( \int_0^1 |f(x) - f_n(x)|^2 \, dx \) need not \( \to 0 \). (Hint: With \( f(x) \equiv 0 \),
a counterexample is provided by taking each \( f_n \) to be an appropriate
multiple of the characteristic function of an appropriate
subset of \([0, 1]\).)

[7]  Let the \( f_n \) and \( f \) belong to \( L_2(0, 1) \), and suppose that \( \|f_n\|_2 \to \|f\|_2 \) while
\( \langle f_n, g \rangle \to \langle f, g \rangle \) for every \( g \in L_2(0, 1) \).
  (a) Show that \( \|f - f_n\|_2 \to 0 \). (Hint: Work as much as possible
with the inner product.)
  (b) Show that unless we assume that \( \|f_n\|_2 \to \|f\|_2 \), the result in (a) need not
follow from the other assumption made. (Hint: Take the \( f_n \) to be (any!)
collection of mutually orthogonal functions in \( L_2(0, 1) \) with \( \|f_n\|_2 = 1 \).)

[8]  Let \( f(x) \geq 0 \) belong to \( L_2(0, \infty) \), and put \( F(x) = \frac{1}{x} \int_0^x f(t) \, dt \), \( x > 0 \).
  (a) Show that
\[ \int_0^\infty (F(x))^2 \, dx = \int_0^1 \int_0^\infty F(x)f(xs) \, dx ds, \]
justifying your work. (Hint: In the formula for \( F(x) \), make the substitution \( s = \frac{t}{x} \).)
  (b) Hence show that if \( \int_0^\infty (F(x))^2 \, dx \) is finite, \( \|F\|_2 \leq 2\|f\|_2 \). (Hint: Apply
Schwarz’s inequality to \( \int_0^\infty F(x)f(xs) \, dx \), and express \( \sqrt{\int_0^\infty (f(xs))^2 \, dx} \) in terms of
\( \|f\|_2 \) and \( s \).)
GEOMETRY AND TOPOLOGY
Answer two of the three questions [9], [10] and [11].

[9] (a) Define what it means for a covering space to be regular.

(b) Let \( B \) be the “bouquet of two circles” formed by identifying two circles along a point. Sketch two covering spaces of \( B \) whose groups of covering space automorphisms (also called “deck transformations”) are infinite cyclic. One of these covers should be regular, the other should not be regular.

[10] Prove that a compact Hausdorff space is regular.

[11] Show that the group \( O(3, \mathbb{R}) = \{X \in GL(3, \mathbb{R}) \mid XX^t = \text{Id}\} \) of three-by-three real orthogonal matrices is a \( C^\infty \) manifold of dimension three.