MCgill University
Department of Mathematics and Statistics
Part A Examination
Pure β

Date: Friday, May 10, 1996
Time: 13:00 - 17:00

Instructions: The SEVEN best answers will account for your final grade.

1. (a) Prove there is no simple group with 48 elements.
   (b) Let $G$ be the group of isometries of the integers, viewed as a metric space with
the usual metric, $d(m,n) = |m - n|$. Show that $G$ is generated by two elements
of order 2 and that, in fact, $G \cong \mathbb{Z}_2 \ast \mathbb{Z}_2$, the free product (coproduct)
of two cyclic groups of order 2.

2. Let $R$ be a ring with unit (possibly non-commutative). If $M$ is a 2-sided $R$-module,
an additive function $d : R \rightarrow M$ is called a derivation if $d(xy) = x(dy) + (dx)y$ for all
$x, y \in R$. Denote the set of all derivations by $\text{Der} (R, M)$.
   (a) Show that $\text{Der}(R, M)$ is an abelian group with pointwise addition. Show that
$\text{Der}(R, -)$ is then a functor from the category of 2-sided modules to abelian
groups.
   (b) Show that this functor is representable. (Hint: the representing object is a quo-
tient of $R \otimes R$, mod a certain submodule.)

3. Let $F = \mathbb{C}[t]$, where $t$ is transcendental over the field $\mathbb{C}$ of complex numbers and let
$K = \mathbb{C}[u]$, where $u = t^3 + t^{-3}$. Show that $F$ is a Galois extension of $K$ and determine
its Galois group.

4. Let the functions $f_n$ and $f$ be Lebesgue measurable on $I = [0, 1]$. Suppose that

$$f_n(x) \rightarrow f(x) \text{ a.e., } x \in I,$$

and

$$\int_{I} |f_n(x)|^2 dx \leq M < \infty \text{ for all } n.$$

(a) Show that $\int_{I} |f(x)|^2 dx \leq M$. 
4. (b) Given \( \varepsilon > 0 \), show that, for \( A \) and \( N \) large enough, there is a Lebesgue measurable \( E \subseteq I \) with \( \text{meas.}(I \sim E) < \varepsilon \) and \( |f_n(x)| \leq A \) for all \( n \geq N \) when \( x \in E \).
   (Hint: You need first to look at \( f \).)

   (c) Hence show that \( \int_I |f(x) - f_n(x)| \, dx \to 0 \). (Hint: If \( E \) is the set from (b), how large can \( \int_{I \sim E} |f(x) - f_n(x)| \, dx \) be?)

5. Let the simply connected domain \( D \subseteq \mathbb{C} \) have more than one boundary point, and suppose that \( 0 \in D \). Consider the family \( F \) of all functions \( f(z) \) analytic in \( D \), with \( f(0) = 0 \) and \( |f(z)| < 1 \), \( z \in D \).

   (a) Show that there are non-constant functions \( f \) in \( F \).

   (b) Given \( z \neq 0 \), \( z \in D \), for which functions \( f \in F \) is \( |f(z)| \) as large as possible? Justify your answer.

6. Let \( \nu \) be a finite complex regular Borel measure on the unit circle (circumference) \( T \), and suppose that

   \[
   \int_T \zeta^n d\nu(\zeta) \to c \quad \text{for } n \to \infty.
   \]

   (a) Show that \( \int_T \zeta^n \phi(\zeta) d\nu(\zeta) \to c \phi(1) \) as \( n \to \infty \) for functions \( \phi \in C(T) \). (Hint: First consider functions \( \phi(\zeta) \) of the special form \( \sum_{n=-N}^{N} A_n \zeta^n \).)

   (b) Assume henceforth that \( c = 0 \). Show that then \( \int_T \zeta^n f(\zeta) d\nu(\zeta) \to 0 \) as \( n \to \infty \) for each \( f \in L_1(T, |\nu|) \), where \( |\nu| \) designates the total variation of \( \nu \).

   (c) Hence show that \( \int_T \zeta^n d|\nu|(\zeta) \to 0 \) as \( n \to \infty \). (Hint: Recall how \( \nu \) is related to \( |\nu| \).)

   (d) Show finally that \( \int_T \zeta^n d\nu(\zeta) \to 0 \) as \( n \to -\infty \) (sic!). (Hint: From (c) we first get, trivially, \( \int_T \zeta^{-n} d|\nu|(\zeta) \) for \( n \to \infty \). See again (b) and the hint to (c).)
7. Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$.

(a) Divide the maps

$$\varphi_1, \varphi_2, \varphi_3, \varphi_4 : S^1 \to S^1$$

$$\varphi_1(x_1, x_2) = (x_1, x_2), \quad \varphi_2(x_1, x_2) = (x_2, x_1),$$

$$\varphi_3(x_1, x_2) = (-x_1, -x_2), \quad \varphi_4(x_1, x_2) = (x_1, -x_2),$$

into homotopy classes. Justify your answer.

(b) Show that any continuous map $\varphi : S^n \to S^1$, $n > 1$, is homotopic to a constant map.

(c) Show that any continuous map $\varphi : S^n \to S^m$, $0 \leq n < m$, is homotopic to a constant map.

8. Let $M = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 | x \cdot x = 1, \ x \cdot y = 0, \ y \cdot y = 1\}$.

(a) Show that $M$ is a regular (imbedded) submanifold of $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$. What is the dimension of $M$?

(b) It is clear that the first projection on $\mathbb{R}^3 \times \mathbb{R}^3$ induces a smooth map $p : M \to S^2$. Show that each $q \in S^2$ has an open neighbourhood $U$ admitting a smooth map $\sigma_u : U \to M$ such that $p \circ \sigma_u = 1_u$. Does there exist a smooth map $r : S^2 \to M$ such that $p \circ r = 1_{S^2}$? Justify your answer.

(c) $M$ is a familiar manifold. Can you identify it?