1. Prove that a group of order 196 contains a normal Sylow-p-subgroup for some p.

2. Let $G$ be the category of groups, $A$ the category of abelian groups, $F$ the category of fields, $D$ the category of integral domains, $M$ the category of $K$-modules ($K$ a commutative ring).

In each of the following cases let $T$ be the appropriate forgetful functor (for example $T : F \rightarrow D$ sends each field $F$ to itself, considered as an integral domain).

Find a left adjoint $S$ for $T$, when $T$ is the following

(a) $T : A \rightarrow G$

(b) $T : F \rightarrow D$

(c) $T : M \rightarrow A$

3. Let $F$ be an algebraic closure of $F_p$, the finite field with $p$ elements. Prove that

(a) $F$ is algebraic Galois over $F_p$ (Galois means that the fixed field of $\text{Aut}_{F_p} F$ is $F_p$ itself).

(b) The map $\varphi : F \rightarrow F$ given by $u \mapsto u^p$ is a nonidentity $F_p$- automorphism of $F$.

(c) The subgroup $H = \langle \varphi \rangle$ is a proper subgroup of $\text{Aut}_{F_p} F$ whose fixed field is $F_p$.

4. Suppose $f$ is an entire function and $|f(z)| \equiv 1$ on $|z| = 1$. Prove $f(z) = cz^n$ for some constant $c$ and integer $n \geq 0$. Hint: First use the maximum and minimum modulus theorems to show

$$f(z) = c \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha_i}z}$$

5. Let $X, \mu$ be a measure space with $\mu(x) < \infty$, and let $\{f_n\}$ be a sequence of $\mu$-measurable functions on $X$. Show that

$$\int_X \frac{|f_n(x)|}{1 + |f_n(x)|} d\mu(x) \rightarrow 0$$

if and only if, for each $\delta > 0$,

$$\mu(\{x \in X; |f_n(x)| > \delta\}) \rightarrow 0.$$
6. For $f \in L_2(\mathbb{R})$ and $\beta > 0$ put $f_\beta(x) = \frac{\beta}{2} \int_{-\infty}^{\infty} e^{-\beta|x-t|} f(t) dt$. Show that $f_\beta \in L_2(\mathbb{R})$ and that $||f_\beta - f||_2 \rightarrow 0$ as $\beta \rightarrow \infty$, justifying your steps.

(Hint: Make first the change of variable $s = x - t$ in above integral, and note also that $f(x) = \frac{\beta}{2} \int_{-\infty}^{\infty} e^{-\beta|s|} f(x) ds$ (sic!).

To estimate $|f_\beta(x)|^2$ and $|f_\beta(x) - f(x)|^2$, note that $e^{-\beta |s|} = (e^{-\beta |s|/2})^2$ and use Schwarz.)

7. Show that a topological space $X$ is Hausdorff iff the image of the diagonal embedding of $X$ into $X \times X$ is a closed set.

8. What are the fundamental groups of the circle, the two dimensional torus, and the figure eight (two circles joined at a point)? What are their homology groups? Exhibit generators for these groups. Give the deRham cohomology of the torus, and again, exhibit generators.