INSTRUCTIONS:

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.
Algebra Module

[ALG. 1]
(a) Assuming the result that for every \( n \) the symmetric group \( S_n \) is the Galois group of some extension of fields, prove that every finite group is a Galois group of some extension of fields.
(b) Show that there is an extension of fields \( F \subset L \) of degree \( n \) and an integer \( a \) dividing \( n \) for which there is no subfield \( F \subset K \subset L \) satisfying \( [K : F] = a \).
(c) Show that there is an extension of fields \( F \subset L \) of degree \( n \) and an integer \( a \) dividing \( n \) for which there is no subfield \( F \subset K \subset L \) satisfying \( [L : K] = a \).

[ALG. 2]
Let \( p \) be a prime. Consider the projective limit of rings
\[
\cdots \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z} \to \cdots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.
\]
The maps are the natural maps \( a \pmod{p^n} \mapsto a \pmod{p^{n-1}} \). It is called the ring of \( p \)-adic numbers and denoted \( \mathbb{Z}_p \).
(a) Show that \( \mathbb{Z}_p \) is a commutative integral domain (and don’t neglect to exhibit its zero and identity element!).
(b) Let \( p \equiv 1 \pmod{4} \). Show that one can solve the equation \( x^2 + 1 = 0 \) in \( \mathbb{Z}_p \).

[ALG. 3]
Let \( \text{Gp} \) be the category of groups and \( \text{AbGp} \) be the category of abelian groups.
(1) Define a functor \( F: \text{Gp} \to \text{AbGp} \), whose effect on objects is \( F(G) = G/G' \) (prove your claims!). Here \( G' \) is the commutator subgroup of \( G \), the subgroup generated by all expressions \( xyx^{-1}y^{-1} \) such that \( x, y \in G \).
(2) Show that \( F \) is not faithful.
(3) Let \( H: \text{AbGp} \to \text{Gp} \) be the natural inclusion functor (so \( H(A) = A \), etc.). Prove that the pair \( (F, H) \) is an adjoint pair of functors. That is, we have natural homomorphisms,
\[
\text{Hom}_{\text{AbGp}}(F(G), A) = \text{Hom}_{\text{Gp}}(G, H(A)) = \text{Hom}_{\text{Gp}}(G, A).
\]

[ALG. 4]
Let \( E/F \) be a finite field extension with \( E = F(\alpha) \), and let \( L \) be an intermediate field, \( F \leq L \leq E \).
(a) Suppose that the minimal polynomial of \( \alpha \) over \( L \) is \( g(x) = \sum_{i=0}^{r-1} b_ix^i + x^r \), and let \( K = F(b_0, \ldots, b_{r-1}) \).
   If \( h \) is the minimal polynomial of \( \alpha \) over \( K \), show that \( g = h \) and conclude that \( L = K \).
(b) Show that there are only finitely many intermediate fields \( L \) between \( E \) and \( F \).
(c) Show that a finite extension has a primitive element iff there are only finitely many intermediate fields.
Let $A, B \subset \mathbb{R}$, and let $A + B = \{a + b : a \in A, b \in B\}$. Prove that if $m(A) > 0, m(B) > 0$, then $A + B$ contains a segment.

**Hint:** There exists $a_0 \in A, b_0 \in B$ where $A$ and $B$ have metric density 1. Choose small $\delta > 0$. Put $c_0 = a_0 + b_0.$ For each sufficiently small $\epsilon$ (positive or negative), define $B_\epsilon$ to be the set of all $c_0 + \epsilon - b$ for which $b \in B$ and $|b - b_0| < \delta$. Then $B_\epsilon \subset (a_0 + \epsilon - \delta, a_0 + \epsilon + \delta)$. If $\delta$ was well chosen and $\epsilon$ sufficiently small, it follows that $A$ intersects $B_\epsilon$, so that $A + B \supset (c_0 - \epsilon_0, c_0 + \epsilon_0)$ for some small $\epsilon_0 > 0$.

**AN. 2**

a) State Riemann-Lebesgue lemma for $L^1(\mathbb{T})$.

Recall that the space $c_0$ consists of all sequences $a_n$ such that $\lim_{|n| \to \infty} a_n = 0$. Prove that the Fourier transform on the torus

$$\mathcal{F}: f \mapsto \{\hat{f}(n)\}_{n \in \mathbb{Z}}$$

maps the space $L^1(\mathbb{T})$ into but not onto $c_0$ by completing the following outline:

b) Prove that $\mathcal{F}(L^1(\mathbb{T})) \subset c_0$. Assume that $\mathcal{F}$ is onto; prove then that there exists $C > 0$ such that

$$||f||_1 \leq C \cdot \sup_n |\hat{f}(n)|.$$  

What theorem have you used?

c) Let $D_k(t) \in L^1[0, 1]$ be the $k$-th Dirichlet kernel. Show that $||D_k||_1 \to \infty$, but $|\hat{D}_k(n)| < M < \infty$ for all $k$ and $n$; get a contradiction with b).

**AN. 3**

a) Define a convolution $f * g$ of two functions $f, g \in L^1(\mathbb{R})$.

b) Let $h = f * g$. Prove that $h$ satisfies $||h||_1 \leq ||f||_1 \cdot ||g||_1$.

**AN. 4**

Given a measurable function $f: X \to \mathbb{R}$ (where $(X, \mu)$ is a measurable space), define a function $F_f: \mathbb{R}_+ \to \mathbb{R}_+$ (the distribution function of $f$) by

$$F_f(t) = \mu\{x : |f(x)| > t\}.$$  

a) Prove Chebyshev’s inequality: for $f \in L^p(X, \mu)$, we have

$$F_f(t) \leq \left(\frac{||f||_p}{t}\right)^p.$$  

b) Let $f = g + h$. Prove that $F_f(t) \leq F_g(t/2) + F_h(t/2)$.
Geometry and Topology Module

[GT. 1]
A symplectic manifold \((M, \Omega)\) is a \(C^\infty\) manifold \(M\) of even dimension \(2n\) endowed with a closed 2-form \(\Omega\) such that \(\Omega^n := \Omega \wedge \ldots \wedge \Omega\) \(n\) times is non-zero at every point of \(M\). Show that if \(M\) is compact, then

i) \(\Omega^n\) is not exact.

ii) \(H_{2k}^d(M) \neq 0\) for all \(k = 1, 2, \ldots, n\).

iii) Show that the sphere \(S^k\) that is symplectic is the 2-sphere \(S^2\).

[GT. 2]
Consider \(\mathbb{R}^4\) endowed with global coordinates \((x, y, p, q)\). Let \(\Omega\) be the 2-form defined by

\[ \Omega = dp \wedge dx + dq \wedge dy. \]

Let \(f: \mathbb{R}^2 \to \mathbb{R}, (x, y) \to f(x, y)\) be a \(C^\infty\) function and let \(F_f\) be the map defined by

\[ F_f(x, y) = (x, y, \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)). \]

i) Show that \(f\) is an immersion.

ii) Show that \(\Omega(X, Y) = 0\) for all \(X, Y\) tangent to the image of \(F_f\).

[GT. 3]
Let \(M\) denote a genus 2 orientable surface.

(a) Prove that \(\pi_1 M\) contains a subgroup isomorphic to a rank 2 free group.

(b) Let \(S\) denote the 2-sphere. Explain why every map \(S \to M\) is null-homotopic.

[GT. 4]
Let \(B\) denote a bouquet of two circles.

(a) Draw (and label) connected degree 4 covering spaces of \(B\) whose automorphism groups are isomorphic to the following groups: (i) \(\mathbb{Z}\); (ii) \(\mathbb{Z}/2\mathbb{Z}\); (iii) \((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})\); (iv) \(\mathbb{Z}/4\mathbb{Z}\).

(b) Let \(\hat{B} \to B\) denote a connected degree 1000 covering space. What is the rank of \(\pi_1 \hat{B}\)?