Department of Mathematics and Statistics  
McGill University

Ph.D. Preliminary Examination, PART A

PURE MATHEMATICS  
PAPER BETA

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Instructions:

1. There are fourteen problems, [1] to [14].  
   Solve three of [1] to [6];  
   solve three of [7] to [10]; and  

2. Pay careful attention to your exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted carefully and in full.

Solve three of the six problems [1], [2], [3], [4], [5] and [6].

[1] (i) Give generators and relations for the dihedral group of order 8, and for the quaternion group.

   (ii) List representatives of all the isomorphism classes of groups of order 8. Show that the groups you list are pairwise non-isomorphic. (There is no need to prove that the list is complete.)

   (iii) Show that the 2-Sylow subgroups of $GL_3(\mathbb{F}_2)$ have order 8, and identify them, up to isomorphism, with one of the groups in (ii). (Hint: there is a simple identification of one specific 2-Sylow subgroup of $GL_3(\mathbb{F}_2)$; find it and use it.)

[2] Let $R$ be a commutative ring with identity.

   (i) Define: (a) $R$ is a PID (principal ideal domain);  
       (b) $R$ is a UFD (unique factorization domain).

   (ii) Show from the definitions that if $R$ is a PID, then $R$ is a UFD. Is the converse always true? Justify your answer.
[3] Let \( S \) be an integral domain, \( R \) a subring of \( S \) (thus, \( R \) is an integral domain too).

(i) Define: \( S \) is an integral extension of \( R \).

(ii) Suppose that \( S \) is an integral extension of \( R \). Show that \( R \) is a field if and only if \( S \) is a field.

[4] Let \( R \) be a ring with identity (possibly non-commutative), let \( M \) a (unitary) module.

(i) Define: \( M \) is projective, resp. injective, over \( R \).

(ii) Let \( R = \mathbb{Z}[\sqrt{-5}] \), and \( M = (u, v) R \), the ideal of \( R \) generated by \( u = 2 \) and \( v = 1 + \sqrt{-5} \). Note that \( M \) is a module over \( R \). Show:

(a) \( M_u \) is free over \( R_u \), \( M_v \) is free over \( R_v \).
(b) \( M \) is not free over \( R \).
(c) \( M \) is projective over \( R \).

(Notation: For a module \( M \) over \( R \), and an element \( u \) of \( R \), \( M_u \) denotes \( S^{-1}M \), where \( S \) is the least multiplicative set containing the element \( u \).)

[5] (i) Let \( R \) be a commutative ring with identity, and let \( M, N \) be \( R \)-modules. Define the tensor product \( M \otimes_R N \) by a universal property.

(ii) Define the concept of co-product of two objects in a category.

(iii) Show that the co-product of two objects exists in the category of commutative rings with identity. Justify your answer.

[6] (i) State, but do not prove, the fundamental theorem of Galois theory. Define the concepts used in the statement.

(ii) Let \( k \) be a field, and let \( \mathbb{K} = k(x_1, \ldots, x_n) \), where \( x_1, x_2, \ldots, x_n \) are algebraically independent over \( k \). Let

\[
f(t) = (t-x_1)(t-x_2)\ldots(t-x_n)
\]

Let \( a_1 \in \mathbb{K} \) be the coefficients of \( f(t) \) so that

\[
f(t) = t^n + a_1 t^{n-1} + \ldots + a_n.
\]

Let \( L_0 \overset{\text{def}}{=} k(a_1, \ldots, a_n) \), and \( L_i \overset{\text{def}}{=} L_{i-1}(x_i) \) for \( i = 1, \ldots, n \).

Show that \( [L_{i+1} : L_i] = n-i \) for all \( i = 0, \ldots, n-1 \).
Solve three of the four problems [7], [8], [9] and [10].

[7] Let \((X, S, \mu)\) be a measure space (where, as usual, \(X \in S\)), let \(f\) be a function \(f : X \rightarrow [0, \infty)\) and suppose that \(f \in L^1(X, S, \mu)\). Define the function \(F : [0, \infty) \rightarrow [0, \infty)\) by

\[
F(a) \overset{\text{def}}{=} \sup_{E \in S} \{ \int f \, d\mu : \mu(E) \leq a \}.
\]

(In words: \(F(a)\) is the supremum of the values of the integral \(\int f \, d\mu\) when \(E\) ranges over \(S\) subject to \(\mu(E) \leq a\).)

Prove that \(F(a)\) is continuous at \(a=0\).

[8] (i) Prove that every finite dimensional normed linear space is a Banach space.

(ii) Prove that if \(V\) is a finite dimensional Banach space, and \(U\) is a closed subspace of \(V\) such that \(U \neq V\), then there is \(v \in V\) such that \(\|v\| = 1\) and \(d(v, U) = 1\).

Note: As usual, we use the term "closed" in the topological sense, and "subspace" in the vector-space sense. \(d(v, U)\) is the distance between \(v\) and \(U\); that is,

\[
d(v, U) = \inf \{ \|v-u\| : u \in U \}.
\]

(iii) Let \(V\) be a normed linear space. Prove that if the unit ball in \(V\) is compact, then \(V\) is finite dimensional. (The unit ball is the set \(\{v \in V : \|v\| \leq 1\}\).
[9] Let \( g: \mathbb{R} \to \mathbb{R} \) and \( F: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be functions. We assume that \( g \) is non-negative and Lebesgue integrable over \( \mathbb{R} \). We assume also that

\[
|F(x, t)| \leq g(x) \quad \text{for all } x \text{ and } t \text{ in } \mathbb{R}.
\]

For every fixed \( x \in \mathbb{R} \), we define the function \( f(x) : \mathbb{R} \to \mathbb{R} \) by

\[
f(x)(t) \overset{\text{def}}{=} F(x, t),
\]

and, for every fixed \( t \in \mathbb{R} \), we define the function \( g(t) : \mathbb{R} \to \mathbb{R} \) by

\[
g(t)(x) \overset{\text{def}}{=} F(x, t).
\]

Further, we assume that conditions (a) and (b) below hold.

(a) \( f(x) : \mathbb{R} \to \mathbb{R} \) is continuous for every \( x \in \mathbb{R} \).

(b) \( g(t) : \mathbb{R} \to \mathbb{R} \) is Lebesgue integrable for every \( t \in \mathbb{R} \).

Prove that the function \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[
f(t) \overset{\text{def}}{=} \int_{\mathbb{R}} f(x, t) \, dx \quad \overset{\text{def}}{=} \int_{\mathbb{R}} g(t) \, dm
\]

is continuous (\( m \) denotes Lebesgue measure).

[10] Let \( (X, S, \mu) \) be a measure space. Assume \( \mu(X) < \infty \). Let \( f, f_n \ (n \in \mathbb{N}) \) be functions in \( L^2(X, S, \mu) \) and suppose that \( \{f_n\} \) converges to \( f \) in the \( L^2 \)-norm. Prove that there is a subsequence \( \{f_{n_k}\} \) of \( \{f_n\} \) with the following property:

for any \( \varepsilon > 0 \), there is a subset \( Y \) of \( X \) such that \( Y \in S, \mu(X - Y) \leq \varepsilon \), and \( \{f_{n_k}\} \) converges to \( f \) pointwise and uniformly on the set \( Y \).
Solve two of the four problems [11], [12], [13] and [14].

[11] Assume that $X$ is a locally compact topological space (recall that this means that $X$ is Hausdorff and every point $x \in X$ has an open neighborhood $U$ such that the closure of $U$ is compact).

(i) Show that every open subset of $X$ with the subspace topology is locally compact.

(ii) Prove the Baire category theorem for locally compact spaces: if $\bigcup_{n \in \mathbb{N}} U_n$ is a dense open set in $X$ for each $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} U_n$ is dense in $X$.

[12] Consider the following surface

(The figure represents an identification space. Each point on one of the two similarly marked segments on the left is identified with a corresponding point on the other segment; and similarly for the segments on the right.)

a) Find a presentation for $\pi_1(S)$.

b) Explain why $S$ is not orientable.

c) Find a double cover (a covering space of degree 2) of $S$ which is orientable; you may draw it as an identification space.

d) Find a connected double cover of $S$ which is not orientable; you may draw it as an identification space.
Let $A$ and $B$ topological spaces, $A$ a subspace of $B$. Assume $A$ is a retract of $B$, meaning that there is a continuous map $f: B \rightarrow B$ such that $f(B) = A$ and $f(a) = a$ for all $a \in A$.

Let $p \in A$. Prove that the inclusion of based spaces $i: (A, p) \rightarrow (B, p)$ has the property that the induced map $i_*: \pi_1(A, p) \rightarrow \pi_1(B, p)$ is injective.

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Let $n$ be any positive integer. Show that the special linear group of $n \times n$ real matrices with determinant 1, in symbols

$$\text{SL}(n, \mathbb{R}) = \{ X \in \text{GL}(n, \mathbb{R}) : \det(X) = 1 \} ,$$

is a Lie group.