McGill University
Department of Mathematics and Statistics
Part A Examination
Pure Mathematics Beta paper

Date: Friday, August 24, 2001
Time: 1:00 p.m. – 5:00 p.m.
Place: Burnside Hall 1234

INSTRUCTIONS

Attempt all eight (8) questions [1] – [8].

Read through each question [k] before attempting any part of it. In each question [k], any part may be attempted even if an earlier part was not done; in doing so, the result of the earlier part may be used.

Do not do things not asked for.

Formulate statements and arguments carefully. The quality of the mathematical writing is an important element of the examination.

Show all calculations.
[1] Let $k$ be a field, and let $\mathbf{V}$ be the category of vector spaces over $k$. As usual, for any $V \in \text{Ob}(\mathbf{V})$, $V^*$ denotes the dual space of $V$, that is, the vector space of all linear functionals $V \to k$.

(i) Prove that there are a functor $E : \mathbf{V}^{\text{op}} \to \mathbf{V}$ and a natural transformation $h : \text{Id}_\mathbf{C} \to E \circ E$ such that $E(V) = V^*$ for all $V \in \text{Ob}(\mathbf{V})$.

[Remarks: The proof should proceed from first principles. $E : \mathbf{V}^{\text{op}} \to \mathbf{V}$ means that $E$ is a contravariant functor from $\mathbf{V}$ to $\mathbf{V}$. The composite $E \circ E$ is then a covariant functor $E \circ E : \mathbf{V} \to \mathbf{V}$.

(ii) Let $h : F \to G$ be a natural transformation between the functors $\mathbf{A} \xrightarrow{F} \mathbf{B}$; $\mathbf{A}, \mathbf{B}$ are arbitrary categories. Show that $h$ has an inverse provided $h_A : F(A) \to G(A)$ has an inverse for each $A \in \text{Ob}(\mathbf{A})$.

(iii) Let $\mathbf{V}_0$ be the full subcategory of $\mathbf{V}$ consisting of the finite dimensional vector spaces as objects. Conclude that $E$, when chosen suitably for (i), restricts to a functor $E_0 : \mathbf{V}_0^{\text{op}} \to \mathbf{V}_0$ so that $E_0 \circ E_0$ is isomorphic to $\text{Id}_{\mathbf{V}_0}$, the identity functor on $\mathbf{V}_0$.

[2] $\mathbb{F}_q$ denotes the finite field of $q$ elements.

(i) Let $p$ be a prime number, $e$ a positive integer, $q = p^e$. Let $f(x) \in \mathbb{F}_p[x]$ be irreducible over $\mathbb{F}_p$. Show that $f(x)$ is a divisor of $x^q - x$ if and only if $\deg(f)$ is a divisor of $e$. State carefully the facts that your proof is based on.

(ii) Factor $x^q - x$ into irreducible factors over $\mathbb{F}_3$; take the underlying set of $\mathbb{F}_3$ to be $\{0, 1, -1\}$.

(iii) With the elements of $\mathbb{F}_9$ being represented by the symbols $\bar{a}$ for $1 \leq i \leq 9$, the addition table for $\mathbb{F}_9$ is a $9 \times 9$ table whose $(i, j)$-entry, for each $1 \leq i, j \leq 9$, gives $\bar{a}_k$ which is the value $\bar{a}_k = \bar{a}_i + \bar{a}_j$. We have the similar multiplication table of $\mathbb{F}_9$.

In each of these two tables, choose and determine 4 particular entries $\bar{a}_k = \bar{a}_i + \bar{a}_j$, respectively $\bar{a}_k = \bar{a}_i \cdot \bar{a}_j$, for which $\bar{a}_i, \bar{a}_j \notin \{0, 1, -1\}$ (to make it non-trivial).
[3] \( \mathbb{Q} \) denotes the additive group, or the field, of the rationals, depending on the context. \( \otimes \) denotes tensor product over \( \mathbb{R} \). \( A, B \) and \( C \) denote \textbf{finitely generated Abelian groups}. \( A_{\mathbb{Q}} \) is the \textit{torsion subgroup} of \( A \), consisting of all \( a \in A \) for which there is \( n \in \mathbb{N} - \{0\} \) such that \( na = 0 \). The \textit{rank} of \( A \) is defined as \( r(A) = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}) \), the dimension of \( A \otimes \mathbb{Q} \) as a vector space over the field \( \mathbb{Q} \).

(i) Prove that if \( A \) is torsion-free (\( A_{\mathbb{Q}} = 0 \)), then \( A \) is free; you may use without proof the fact that a subgroup of a finitely generated free Abelian group is free.

(ii) Prove that \( A \cong A_{\mathbb{Q}} \times \mathbb{R}^r \) where \( r = r(A) \).

(iii) Show that \( \mathbb{Q} \) is a flat \( \mathbb{R} \)-module: for \( A \longrightarrow B \) injective, \( A \otimes \mathbb{Q} \longrightarrow B \otimes \mathbb{Q} \) is injective as well.

(iv) Prove that if \( 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \) is an exact sequence, then \( r(B) = r(A) + r(C) \).

[4] Given the real quantities \( a_n \) and \( \gamma_n \), \( n = 0, 1, 2, \ldots \), suppose that the series (N.B. \textit{not} necessarily a Fourier series!)

\[
(*) \quad \sum_{n=0}^{\infty} a_n \cdot \cos(nx + \gamma_n)
\]

converges to a finite limit for each \( x \in \mathcal{E} \), a Lebesgue measurable subset of \( [-\pi, \pi] \) with \( |\mathcal{E}| > 0 \). (\( |S| \) denotes the Lebesgue measure of a set \( S \subseteq \mathbb{R} \)).

(a) Show that there is a Lebesgue measurable set \( \mathcal{E}_0 \subseteq \mathcal{E} \), \( |\mathcal{E}_0| > 0 \), such that (*) converges \textit{uniformly} for \( x \in \mathcal{E}_0 \).

(b) Show that \( \int_{\mathcal{E}_0} \cos^2(nx + \gamma_n) \, dx \longrightarrow \frac{1}{2} |\mathcal{E}_0| \). (Hint: start by using the half angle formula and then the addition formula for cosine.)

(c) Hence show that, under the given condition on (*), we have \( a_n \longrightarrow 0 \). (Hint: refer to the result in (a).)
Given the Lebesgue measurable functions \( f_n(x) \) and \( f_n(x) \), \( n=1, 2, \ldots \) on \( I=[0,1] \), suppose that \( f_n(x) \xrightarrow{n} f(x) \) a.e. \( x \in I \); and that

\[
\int_0^1 |f_n(x)|^{3/2} \, dx \leq C < \infty
\]

for all \( n \).

(a) Show that \( \int_0^1 |f(x)|^{3/2} \, dx \leq C \).

Fix now a large \( M \), and let, for each \( n \),

\[
E_n = \{ x \in I : |f(x)| \leq M \text{ and } |f_n(x)| \leq M \}.
\]

(b) Show that \( \int_{E_n} |f_n(x) - f(x)| \, dx \xrightarrow{n} 0 \).

(Hint: Taking the characteristic functions \( \chi_{E_n} \) of each \( E_n \), work with the functions \( \chi_{E_n}(x) \cdot (f_n(x) - f(x)) \).

(c) If \( G_n = I \setminus E_n \), show that \( \int_{G_n} |f_n(x) - f(x)| \, dx \leq \frac{4C}{\sqrt{M}} \).

(Hint: First verify that \( \frac{|f_n(x)| + |f(x)|}{|f_n(x)|^{3/2} + |f(x)|^{3/2}} \leq \frac{2}{\sqrt{M}} \) for \( x \in G_n \); for this it is good to note that \( |f_n(x)| + |f(x)| \leq 2 \cdot \max(|f_n(x)|, |f(x)|) \).

(d) Hence show that \( \int_0^1 |f_n(x) - f(x)| \, dx \xrightarrow{n} 0 \).
[6] Evaluate the following two limits:

(a) \[ \lim_{n \to \infty} \int_0^n (1 - \frac{x}{n})^n e^{x/2} \, dx \]

(b) \[ \lim_{n \to \infty} \int_0^n (1 + \frac{x}{n})^n e^{-2x} \, dx \]

Justify your answers fully.

[7] Here, a "manifold" means a paracompact smooth \((C^\infty)\) manifold.

(i) Give the definition of an orientation of a manifold.

(ii) Choose and prove either the "if" or the "only if" part of the theorem that states that an \(n\)-dimensional manifold has an orientation if and only if there exists a non-vanishing smooth \(n\)-form on it. (Facts on partitions of unity can be freely used.)

(iii) Illustrate the above on \(S^2\), the 2-sphere, by exhibiting both an orientation and a non-vanishing smooth 2-form on it.

[8] Let \((\tilde{X}, \tilde{p})\) be a covering space of \(X\); let \(x \in X\) and \(\tilde{x} \in \tilde{X}\) such that \(\tilde{p}(\tilde{x}) = x\). We have the induced homomorphism \(p_* : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, x)\).

(i) Show that \(p_*\) is injective, and that there is a bijection between the set \(p^{-1}(\{x\})\) and the set of all cosets of the image of \(p_*\) in \(\pi_1(X, x)\). For the proof, the Covering Homotopy Theorem may be used, which need not be proven.

(ii) Describe how the image of \(p_*\) in \(\pi_1(X, x)\) changes when we vary \(\tilde{x}\) in \(p^{-1}(\{x\})\).

(iii) Prove that \(\pi_1(\mathbb{P}^2)\), the fundamental group of the real projective plane, is cyclic of order 2.

(iv) Give a glueing model (plane figure with identification data) for \(\mathbb{P}^2\), and in it, a generator for \(\pi_1(\mathbb{P}^2)\).