Partial Differential Equations

1. An elastic membrane subject to uniform gas pressure satisfies the equation

\[ \psi_{tt} + P = a^2 \nabla^2 \psi, \]

where \( a^2 \) and \( P \) are constants.

Find the displacement of a circular membrane of radius \( b \), if it is clamped on the circumference, i.e. \( \psi(b, t) = 0 \).

The initial displacement is \( f(r) \) and the initial velocity is \( g(r) \).

2. Obtain the Green's function and then solve the general Neumann problem for a semi-infinite bar

\[ a^2 \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial t} = h(x, t); \quad 0 < x < \infty, \quad t > 0 \]

with (i) \( \psi(x, 0) = f(x) \) and (ii) \( \psi_x(0, t) = g(t) \). You may assume that \( a^2 \) is constant.

3. Solve

\[ \psi_{xx} + \psi_{yy} = 0; \quad x > 0, \quad y > 0 \]

\[ \psi(0, y) = T_0 \]

\[ \psi_y(x, 0) = h[\psi(x, 0) - T_1] \]

\[ \lim_{y \to \infty} \psi(x, y) = T_0, \]

and interpret physically.
USEFUL INFORMATION

1. Laplacian in cylindrical coordinates.

\[ \nabla^2 \psi(r, \theta, z) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}. \]

2. Laplacian in spherical coordinates.

\[ \nabla^2 \psi(\rho, \phi, \theta) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial \psi}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 \psi}{\partial \theta^2}. \]

3. \[ \int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2\delta_{nm}}{2n + 1}, \text{ where } P_n(x) \text{ and } P_m(x) \text{ are Legendre polynomials.} \]

4. \[ \int_{0}^{b} xJ_n(\lambda_n x)J_n(\lambda nt x)dx = \frac{b^2}{2} [J_{n+1}(\lambda nt b)]^2 \delta_{kt}, \text{ where } J_n(x) \text{ are Bessel functions of } \text{ the first kind of order } n, \text{ and } J_n(\lambda b) = 0 \text{ for all } \lambda. \]
OPTIMIZATION

4. A linear programming model that maximizes the profit of a company, subject to labor and facility restrictions, is

\[
\begin{align*}
\text{Max} & \quad 15x_1 + 10x_2 + 6x_3 \\
\text{s.t.} & \quad 10x_1 + 8x_2 + 6x_3 \leq 50 \\
& \quad 5x_1 + 12x_2 + 8x_3 \leq 35 \\
& \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.
\end{align*}
\]

\((x_i, \ i = 1, 2, 3\) are investment dollars.\)

(i) Prove that \(x_1^* = 4, \ x_2^* = 1.25, \ x_3^* = 0\) is an extreme point of its feasible set.

(You may relate to a theorem.)

(ii) Find an optimal solution and the shadow prices.

(iii) Suppose that the technological coefficient \(a_{12} = 8\) is increased by \(\Delta a_{12} = 0.2\). By how much should the manpower coefficient \(b_1 = 50\) change in order to keep the maximal profit at the same level? Answer this question using sensitivity analysis.

5. A differentiable convex function \(f : \mathbb{R}^n \to \mathbb{R}\) is said to have the LFS ("locally flat surface") property at \(x^* \in \mathbb{R}^n\) if

\[
N(\nabla f(x^*)) = D_f^\infty(x^*),
\]

Here

\[
N(\nabla f(x^*)) = \{d \in \mathbb{R}^n : \nabla f(x^*) \cdot d = 0\}
\]

is the null-space of the gradient and

\[
D_f^\infty(x^*) = \{d \in \mathbb{R}^n : f(x^* + \alpha d) = f(x^*), \ 0 \leq \alpha \leq \overline{\alpha}, \ \text{ for some } \overline{\alpha} > 0\}
\]

is the cone of directions of constancy of \(f\) at \(x^*\). Consider a program

\[
\begin{align*}
\text{(C)} \quad \text{Min} & \quad f(x) \\
\text{s.t.} & \quad f^i(x) \leq 0, \quad i \in \mathcal{P} = \{1, \ldots, m\}
\end{align*}
\]

where all functions are convex and differentiable, defined on all of \(\mathbb{R}^n\). Given a feasible point \(x^*\) of (C).
(i) Prove the following claim: "If all constraints have the LFS property at \( x^* \) then \( x^* \) is an optimal solution of (C) if, and only if, the Karush-Kuhn-Tucker condition is satisfied at \( x^* \)."

(ii) Using the above claim prove that the Karush-Kuhn-Tucker condition is both necessary and sufficient that a feasible point of a convex program, with only linear constraints, be optimal.

(iii) Give an example in one variable showing that the Karush-Kuhn-Tucker condition may not be satisfied at an optimal solution if the constraints are not LFS. How do the characterizations of optimality look in this case?

6. Consider a special case of the classic navigation problem of Zermelo: A boat, situated at the "origin" at time \( t = 0 \), is moving with a velocity \( v \) of unit magnitude relative to a stream of constant speed \( V = 2 \). The problem is to determine a constant steering angle \( \theta \) (relative to \( x_1 \)) that minimizes the time \( t \) required to reach a target, say, the disc

\[
T = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : (x_1 - 5)^2 + (x_2 - 1)^2 \leq 1 \right\}.
\]

Note that the dynamics is described by the system of differential equations

\[
\frac{dx_1}{dt} = 2 + \cos \theta
\]
\[
\frac{dx_2}{dt} = \sin \theta.
\]

(i) Formulate the problem as a mathematical program.

(ii) Describe a general class of numerical methods that you can use to solve the problem. Starting from \( \theta^0 = 30^0 \) and \( t^0 = 1.41 \) find a better approximation to the optimal solution.

(iii) How would you verify whether your numerical solution is a local or a global optimum?

(iv) How does the mathematical program look if \( \theta \) is not constant but it is allowed to change finitely many times before the boat reaches the target \( T \)? How does a continuous optimal control formulation of the problem look? How would you check optimality of \( \theta = \theta(t) \) in the latter case?
NUMERICAL ANALYSIS

7. (a) Let \( \overline{G} \) map \( \mathbb{R}^m \to \mathbb{R}^m \), and be such that for a closed convex set \( B \subset \mathbb{R}^m \), \( \overline{G}(B) \subset B \). What other condition on \( \overline{G}(\overline{X}) \) in terms of the vector-norm \( \| \overline{X} \|_\infty \) will ensure that the sequence \( \{ \overline{X}_n \}, \overline{X}_{n+1} = \overline{G}(\overline{X}_n), \overline{X}_0 \in B \), converges to a limit point \( \overline{Z} \in B \)?

(b) Let \( K = \max_{\overline{X} \in B} \| J(\overline{G}) \|_\infty \), where \( J(\overline{G}) \) is the Jacobian of \( \overline{G} \). Give an upper bound for \( K \) in terms of the partial derivatives \( \frac{\partial g_j}{\partial x_i} \), where \( \overline{G}(\overline{X}) = \{ g_j(x_1, x_2, \cdots, x_m) \} \), \( j = 1, \cdots m \); justify your answer. You may use the formula for \( \| A \|_\infty \) (\( A \) a square matrix) in terms of the absolute row-sums.

(c) Find a solution to the system (to 4 decimals):

\[
\begin{align*}
    x^4 + y &= 100 \\
    \ln(x) + e^y &= 10 
\end{align*}
\]

in the positive quadrant, and justify the convergence.

8. (a) Describe how you would use Euler's method to give an approximate solution of the initial value problem

\[
\begin{align*}
    \frac{d\overline{x}}{dt} &= \overline{F}(\overline{x}, t) \\
    \overline{x}(0) &= \overline{x}_0 \quad (\overline{x}_0 \in B)
\end{align*}
\]

at the points \( t_n = nh \in [0, K] \).

(b) Now consider the scalar system \( \frac{dx}{dt} = f(x, t) \) with \( x(0) = x_0 \) and \( f \in C^k[0, K] \). Show how to find the Taylor approximation \( x_1 \) of order \( k \) to \( x(h) \) and show that \( |x_1 - x(h)| = O(h^{k+1}) \).

(c) Why are Euler's and Taylor's algorithms not used in practice?

(d) Define how a Runge-Kutta method of \( m \) stages and order \( k \) is defined, and illustrate by finding all the \( RK \) methods of 2 stages and order 2. Justify that if the step-size is \( h \) then the truncation error/step is \( O(h^3) \).
(e) Illustrate by estimating the value of \( x(0.1) \) given by

\[
\frac{dx}{dt} = t + x^2, \quad x(0) = 0
\]

(f) In the specific problem in (e), can you find a much more accurate way of solving the problem?

9. Consider the initial and boundary value problem

\[
U_x = U_{yy}, \quad (0 < y < 1, \quad x > 0);
\]

\[
U(0, y) = 1; \quad U_y(x, 0) = 0; \quad U_y(x, 1) = 1.
\]

(a) Using the classic explicit scheme, and employing the control difference for B.C's, write the finite difference equations for \( U_{r,s}(r,s, = 0,1,2, \ldots N) \) in a matrix form.

(b) In terms of matrix stability analysis, discuss the stability property of the above scheme.