McGILL UNIVERSITY

DEPARTMENT OF MATHEMATICS AND STATISTICS

APPLIED MATHEMATICS PART BETA EXAMINATION

Date: May 4, 2009.
Time: 1:00 P.M. - 5:00 P.M.

INSTRUCTIONS

1. This exam consists of 3 modules:

2. Each module consists of 4 questions.

3. Students will be assessed on answers to 7 questions.
   At most three questions may be taken from each of the three modules.
Numerical Analysis

1. Suppose $A \in \mathbb{C}^{m \times n}$ has rank $r$.
   (i) State the singular value decomposition (SVD) of $A$.
   (ii) Show how the SVD can be used to find all solutions to
        $$\min_x \|b - Ax\|_2.$$ 
   (iii) Under which condition is there a unique solution in part (b)?

2. Given a nonsingular matrix $A \in \mathbb{C}^{n \times n}$, suppose you would like to obtain a unitary matrix $Q$ and an upper-Hessenberg matrix $H$ such that
   $$AQ = QH.$$ 
   (i) Design an algorithm to compute $Q$ and $H$ iteratively, column by column. Start with an arbitrary nonzero vector $b \in \mathbb{C}^n$, and let the first column of $Q$ be $q_1 = b/\|b\|_2$.
   (ii) Explain how the algorithm GMRES uses the above to solve $Ax = b$ iteratively.
3. (i) Show that the exact solution of the differential equation $\dot{u} = tp$ satisfies

$$u(t_{n+1}) = u(t_n) + \sum_{j=1}^{p+1} \frac{p!}{j!(p-j+1)!} h^j t_n^p - j + 1,$$

where $h = t_{n+1} - t_n$.

(ii) In the remaining parts of this question we only consider $s$-stage Runge-Kutta methods with Butcher tableaus $\begin{array}{c|c} c \end{array} \begin{array}{c} A \end{array} \begin{array}{c} b^T \end{array}$ which satisfy the condition that $c_i = \sum_{j=1}^s a_{ij}$. In a sentence, why is this a natural condition to impose?

(iii) Using (i) and (ii) show that necessary conditions for a Runge-Kutta method to be of order $p$ are

$$\sum_{i=1}^s b_i c_i^{j-1} = \frac{1}{j}, \quad j = 1, \ldots, p.$$ 

(iv) Given that the conditions found in (iii) are necessary and sufficient for $p = 1$ and $p = 2$ and that the only additional condition for a general Runge-Kutta method to be order 3 is $\sum_{i=1}^s \sum_{j=1}^s b_i a_{ij} c_j = 1/6$, now restrict attention to symplectic Runge-Kutta methods which are characterized by

$$b_i a_{ij} + b_j a_{ji} - b_i b_j = 0, \quad \forall \ 1 \leq i \leq s, \ 1 \leq j \leq s$$

and

(a) show that every consistent symplectic Runge-Kutta method also satisfies $\sum_{i=1}^s b_i c_i = 1/2$, and so is of order at least two,

(b) every consistent symplectic Runge-Kutta method which satisfies $\sum_{i=1}^s b_i c_i^2 = 1/3$ is of at least order 3,

(c) hence construct a two-stage symplectic Runge-Kutta method with order 3 and $b_2 = 3/4$. [Hint: You may find it convenient to start from the Butcher tableau

$\begin{array}{c|ccc} c_1 & a_1 & c_1 - a_1 \\ c_2 & a_2 & a_2 \\ \hline b_1 & b_2 \end{array}$

and begin by enforcing consistency and symplecticity]
4. Consider the PDE $\Delta u(x, y) = f(x, y)$, $(x, y) \in \Omega = [0, 1] \times [0, 1]$, with Dirichlet boundary conditions, and let $\Delta_h$ denote the standard five point finite difference discretization on the regular mesh $x_i = ih$, $y_j = jh$, $h = 1/(N + 1)$, let $U_{ij}$ be the numerical solution at $(x_i, y_j)$ and let $u_{ij} = u(x_i, y_j)$ be the solution of the original PDE at the same point.

(i) Let $\|U_{ij}\|_\infty = \max_{ij} |U_{ij}|$ and show that $\|\Delta_h(u_{ij} - U_{ij})\|_\infty \leq Mh^2$, under suitable assumptions on the solution $u(x, y)$, which you should state.

(ii) Show that $\Delta_h$ satisfies a maximum principle; that is, if $\Delta_h\psi_{ij} \leq 0$ then $\psi_{ij}$ attains its minimum on $\partial \Omega$, and if $\Delta_h\psi_{ij} \geq 0$ then $\psi_{ij}$ attains its maximum on $\partial \Omega$.

(iii) Let $v_{ij} = \frac{1}{4} \left[ \left(x_i - \frac{1}{2}\right)^2 + \left(y_j - \frac{1}{2}\right)^2 \right]$ and show that $\|v_{ij}\|_\infty = 1/8$ and $\Delta_h v_{ij} = 1$.

(iv) Show that

$$\Delta_h \left[ \psi_{ij} - \|\Delta_h\psi_{ij}\|_\infty v_{ij} \right] \leq 0$$

and hence, if $\psi$ vanishes on $\partial \Omega$, that

$$-\frac{1}{8}\|\Delta_h\psi_{ij}\|_\infty \leq \psi_{ij}$$

for all points on the mesh.

(v) Hence deduce that $\|u_{ij} - U_{ij}\|_\infty \leq \frac{1}{8}Mh^2$.

(vi) Suppose the boundary conditions are changed to Neumann conditions $u(0, y) = a(y)$ on the edge $x = 0$ of the box. How should the discretization be modified to accommodate this whilst maintaining accuracy?
Partial Differential Equations

1. Find the solution \( u = u(x, t) \) of

\[
2xtu_x + u_t = u, \quad u(x, 0) = f(x)
\]

where \( f \in C^1(-\infty, \infty) \) and prove that when \( f(0) = 0 \) one has \( \lim_{t \to \infty} u(x, t) = 0 \).

2. Assume the fact that on \( \mathbb{R}^3 \) \( \Delta_1 \frac{1}{r} = -4\pi \delta \) in the distribution sense where

\[
r = (x_1^2 + x_2^2 + x_3^2)^{-1/2}.
\]

Now consider the operator \( Lu = a_1^2 \partial_1^2 u + a_2^2 \partial_2^2 u + a_3^2 \partial_3^2 u \) and the function \( E(x_1, x_2, x_3) = (x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2)^{-1/2} \) where \( a_1, a_2 \) and \( a_3 \) are positive constants and \( \partial_i \) denotes partial differentiation wrt \( x_i \). Prove that

\[
LE = -4\pi a_1 a_2 a_3 \delta
\]

in the distribution sense.

3. Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary. Prove that there is a constant \( C > 0 \) depending only on \( \Omega \) such that

\[
\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega).
\]

4. Let \( \{u_k(x, t)\} \) be a monotone sequence of solutions of heat equation \( u_t - \Delta_x u = 0 \) in \( \mathbb{R}^n \times (0, \infty) \). Suppose there is a point \( (x_0, t_0) \in \mathbb{R}^n \times (0, \infty) \) such that \( \{u_k(x_0, t_0)\}_{k=1}^\infty \) is bounded. Prove that for any compact subset \( K \subset \mathbb{R}^n \), any \( T \geq 0 \), the sequence \( \{u_k(x, t)\}_{k=1}^\infty \) is uniformly convergent in \( K \times [t_0, t_0 + T] \), and the limit function satisfies the heat equation.
Probability Module

1. (a) Let $Ω$ be an uncountable set, and let $F = \{ A, A^c : A \text{ is a countable subset of } Ω \}$. Show that $F$ is a $σ$-algebra, and that

$$P(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{if } A \text{ is uncountable,} \end{cases}$$

defines a probability on $F$.

(b) Suppose that $\{X_n, n \geq 1\}$ is a sequence of non-negative r.v.’s on a probability space $(Ω, F, P)$, where $Ω$ is finite and $F$ is the family of all subsets of $Ω$. Show that if $X_n \to 0$ in probability, then $X_n \to 0$ almost surely.

2. (a) Suppose that $X_1, X_2, \ldots$ are independent r.v.’s, and that $X_n$ has density function

$$f(x) = \begin{cases} \lambda_n e^{-\lambda_n x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

where $\lambda_n > 0$ for each $n$. Show that $\sum_{n=0}^{\infty} X_n = \infty$ a.s. if $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$. (Hint: Compute $Ee^{-X_n}$ and use the fact that $\prod_{i=1}^{n} (1 + a_i) \geq \sum_{i=1}^{n} a_i$ for non-negative $a_i$’s.)

(b) Let $(Ω, F, P)$ be a probability space. Define what is meant by an atom $A \in F$. Let $A \in F$ be an atom and suppose $X$ is a random variable on $(Ω, F, P)$ which is a.s. finite on $A$. Show that there is $k \in \mathbb{R}$ such that $X = k$ a.s. on $A$.

3. (a) Define what is a supermartingale, submartingale, and martingale.

(b) State and prove Doob’s optional sampling theorem.

(c) Let $Y_1, Y_2, \ldots$ be i.i.d. with finite mean $µ$, and let $X_n = \sum_{i=1}^{n} Y_i$. Let $T$ be a finite stopping time for $\{X_n, n \geq 1\}$. Show that:

i. if all $Y_i > 0$ then $EX_T = µET$,

ii. if $ET < \infty$, then $E|X_T| < \infty$ and $EX_T = µET$.

(Hint: let $T_n = T \wedge n$ and use Doob’s theorem to prove (i). Use (i) to prove (ii).)

4. (a) State and prove the Helly selection theorem.

(b) Suppose that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are random variables defined on the same probability space, with values in $\mathbb{R}$. Assume that $X_n \overset{d}{\to} X$ and $X_n - Y_n \overset{P}{\to} 0$. Prove that $Y_n \overset{d}{\to} X$. 