INSTRUCTIONS

1. This exam consists of 3 modules:
   Optimization, Numerical Analysis and Applied PDE

2. Each module consists of 4 questions.

3. Students will be assessed on answers to 7 questions.
   At most three questions may be taken from each of the three modules.
Optimization Module

Problem 1) (a) Give a brief but careful description of a Linear Conjugate Direction method.

(b) Explain what a Krylov subspace of dimension k is for a vector \( r_0 \in \mathbb{R}^n \). What is the significance of such subspaces with regards to analyzing performance of the Linear Conjugate Gradient Method.

(c) What is a Nonlinear Conjugate Gradient Method? Briefly explain how to obtain such a method by adapting a Linear Conjugate Gradient.

Problem 2) (a) Give a brief description of Newton’s method for solving a system of nonlinear equations.

(b) We wish to solve the linear program \( \min c^T x \) subject to \( Ax = b, x \geq 0 \). Here \( c, x \) are vectors in \( \mathbb{R}^n \), \( b \in \mathbb{R}^m \) and \( A \) is an \( m \times n \) matrix of full row rank. Define the dual problem.

(c) Give necessary conditions for any primal-dual pair of optimal solutions. Is this condition sufficient for optimality of the pair? Explain.

(d) Briefly explain a primal-dual interior method approach for solving the linear program in question. In other words, how may one employ Newton’s method for nonlinear equations for finding optimal solutions for the primal and dual problems.

Problem 3) State and prove Edmond’s Arborescence Packing Theorem.

Problem 4) (a) Define the following terms: network matrix, polytope and vertex (or extreme point) of a polytope.

(b) An \( n \times n 0,1 \) matrix \( A \) has the consecutive ones property if for each column \( k \), there exists \( 1 \leq i \leq j \leq n \) such that the only 1’s in column \( k \) are \( A_{ik}, A_{(i+1)k}, \ldots, A_{jk} \). Show that for such a matrix \( A \) and integer vector \( b \in \mathbb{R}^n \), the region \( P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} \) has integral extreme points.

(c) Suppose we have integer bounds \( l_i \leq u_i \) for each \( i \in \{1, 2, \ldots, n\} \). If \( A \) has the consecutive ones property, explain why

\[
\max w \cdot x : Ax \leq b, l_i \leq x_i \leq u_i \quad \text{for all } i = 1, 2, \ldots n
\]

always has an integral optimal solution if \( w, b \in \mathbb{Z}^n \).
Applied PDE Module

Problem 1) Consider the problem

\[-\nabla \cdot [a(x) \nabla u(x)] = f(x) \quad \text{for } x \in \Omega,\]
\[\nu \cdot \nabla u(x) = g(x) \quad \text{for } x \in \partial \Omega.\]

where \(\Omega\) is a bounded domain with smooth boundary, \(\nu\) is the normal to the boundary and \(a(x)\) is a positive differentiable function on the closure of \(\Omega\).

(a) State a necessary condition for this problem to have a solution.
(b) Assuming (a), prove that the above problem has at most one solution in \(C^2(\Omega)\).
(c) Use the divergence theorem to motivate and then carefully specify the weak formulation of this problem.
(d) Assuming the condition in (a), briefly outline the proof of the existence of weak solutions.

Problem 2) (a) Derive formulae for the solutions \(u(x, t)\) and \(v(x, t)\) of the system

\[
\frac{\partial u}{\partial t} + 3 \frac{\partial v}{\partial x} = 0, \quad \text{subject to } u(x, 0) = u^0(x),
\]
\[
\frac{\partial v}{\partial t} + 3 \frac{\partial u}{\partial x} = 0, \quad \text{subject to } v(x, 0) = v^0(x).
\]

(b) Evaluate the solutions at \((x, t) = (1, 1)\) and \((x, t) = (2, 0.5)\) in the case that

\[
u^0(x) = \begin{cases} 
0 & \text{for } x < 0 \\
2 & \text{for } x \geq 0 
\end{cases}, \quad \text{and } v^0(x) = \begin{cases} 
2 & \text{for } x < 1 \\
4 & \text{for } x \geq 1 
\end{cases}.
\]

(c) Note that the solutions are identically constant over certain regions in the \((x, t)\)-plane. Sketch these regions. Verify that the Rankine-Hugoniot condition holds across the boundary between the regions containing \((x, t) = (1, 1)\) and \((x, t) = (2, 0.5)\).
**Problem 3** The forced vibration of an infinitely long uniform beam is described by

\[ \frac{\partial^4 u}{\partial x^4} + \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{1}{EI} P(x, t), \]

where \( u(x, t) \) is the lateral deflection of the beam, and \( c, E, I \) are constants. The forcing \( P(x, t) \) is localized and impulsive, i.e.,

\[ P(x, t) = \delta(x)\delta(t) \]

where \( \delta \) denotes the Dirac measure.

Use Fourier transforms to find \( u(x, t) \), given that \( u(x, 0) = 0 \). Evaluate all the integrals, and interpret the results physically.

You may use the identity

\[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\xi^2 a - i\xi x) d\xi = \frac{1}{\sqrt{2a}} \exp(-x^2/4a), \]

where \( a \) can be complex.

**Problem 4** Consider a solid sphere \( \Omega \subset \mathbb{R}^3 \) of radius \( a \), centered at the origin. Let the sphere have constant mass density \( \mu \). By Newton's law, the gravitational force \( \vec{F} \) exerted by this sphere on a unit mass at point \( \xi \) is given by

\[ \vec{F}(\xi) = c \int_{\Omega} \frac{\mu(x - \xi)}{|x - \xi|^3} dx. \]

Associated with this force is a gravitational potential \( u \), which is harmonic outside \( \Omega \), vanishes at infinity, is \( C^1(\mathbb{R}^3) \), and satisfies

\[ \Delta u = \nabla \cdot \vec{F} \text{ in } \Omega. \]

(a) Evaluate the gravitational potential, which is defined by

\[ u(\xi) := c \int_{\mathbb{R}^3} \frac{\mu}{|x - \xi|} dx. \]

(b) Evaluate the force \( \vec{F} \), by first showing that \( \vec{F} = \nabla u \).
Numerical Analysis

Problem 1) Consider the advection equation

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = 0,$$

where $u > 0$ is a constant. To keep things simple, we use uniform grids in time and space, i.e. $x = j \Delta x$ and $t = n \Delta t$. One possible finite difference formula would be

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + u \frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0,$$

(1)

where $\phi_j^n$ refers to the $n$th timestep at the $j$th grid point in $x$.

(a) Assuming $\phi$ is sufficiently smooth, calculate the leading terms of the truncation error of the scheme (1) and state its order in time and space.

(b) Explore the stability of (1) in terms of the Courant number, $C = u \Delta t / \Delta x > 0$ using the energy method. Here's how: If

$$\|\phi^n\|_2 = \sum_j (\phi_j^n)^2,$$

then stability requires that

$$\|\phi^{n+1}\|_2 \leq \|\phi^n\|_2 \leq \|\phi^0\|_2.$$

You need to determine when this is true in terms of $C$.

Hint: Assume periodic boundary conditions in $x$.

Problem 2) Consider the equation

$$x = \frac{1}{b} e^{-x},$$

(2)

where $e = \exp(1)$, and $b > 1$ is a constant.

(a) Using Banach's fixed point theorem, show that (2) has exactly one solution $\overline{x}$ in $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. (Hint: prove that there is exactly one solution in the closed interval $[0, a]$ for any $a \geq 1$).

(b) Use a Taylor expansion to find the order of convergence close to the zero $\overline{x}$.

(c) Write down Newton's method for $g(x) = x - \frac{1}{b} e^{-x}$. Illustrate graphically that both schemes – the fixed point method and the Newton method – with starting value $x_0 = 1$ for both schemes, converge to the solution $\overline{x}$.
Problem 3) Consider the family of finite-difference schemes for the 2nd derivative that can be written as
\[
\frac{\delta^2 \phi_i}{\delta x^2} = \frac{1}{\Delta x^2} \sum_j W_j \phi_{i+j},
\]
where \( \delta^2 \phi_i/\delta x^2 \) is the numerical approximation to \( \partial^2 \phi/\partial x^2 \) at \( x = x_i \) and the \( W_j \), determine the scheme (e.g., \( W_{-1} = W_1 = 1 \) and \( W_0 = -2 \) ought to be familiar).

(a) Show that
\[
\frac{\delta^2 \phi_i}{\delta x^2} = \frac{\partial^2 \phi}{\partial x^2} \Bigg|_{x_i} + E, \quad E := \sum_{n=3}^{\infty} \alpha_n \Delta x^{n-2} \frac{\partial^n \phi}{\partial x^n} \Bigg|_{x_i}
\]
where \( E \) is the error. with \( \alpha_n = \frac{1}{n!} \sum_j j^n W_j \), \( \alpha_0 = \alpha_1 = \alpha_2 = 0 \), and the lowest non-zero \( \alpha_n \) \( (n \geq 3) \) determines the accuracy of the scheme.

(b) Use the results of a) in the heat diffusion equation
\[
\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}.
\]
Assume there are no time truncation errors and use a Fourier series to show that even-order-in-space schemes yield amplitude errors while odd-order-in-space schemes yield dispersion errors. Hint: Fourier \( \phi(x, t) = \sum_k \hat{\phi}_k(t) e^{ikx} \).

Problem 4) Consider the matrix \( A_{m,\alpha} \) resulting from a finite difference discretization.
\[
A_{m,\alpha} = \begin{bmatrix}
4 & -1 & & \\
-1 & 4 & -1 & \\
& \cdots & \cdots & \cdots \\
& & -1 & 4 & -1 \\
& & & -1 & 4 \\
& & & & -1 & 4 \\
\end{bmatrix} + \alpha I_m \in \mathbb{R}^{m \times m}.
\]
Here, \( m \geq 1 \) is an integer, \( \alpha \in \mathbb{R} \) is constant, and \( I_m \) denotes the identity matrix in \( \mathbb{R}^{m \times m} \).

(a) Use the Cholesky decomposition to find all \( \alpha \in \mathbb{R} \) for which \( A_{2,\alpha} \) (i.e. \( m = 2 \)) is SPD.
(b) Choose a *simple* iterative scheme that can solve the linear system

$$A_{0,m}x = b,$$

where $b \in \mathbb{R}^m$, approximately. Justify your choice. Furthermore, write a short Matlab code that implements this method for the given matrix $A_{0,m}$ (input: $b$, output: $x$); make use of the sparse structure of $A_{m,0}$, and implement matrix-vector multiplications explicitly.