This exam consists of 3 sections with 4 problems each: Optimization, Applied PDE and Numerical Analysis. Attempt 6 out of these 12 problems. Of these, three must be from one section, and the remaining three from another section.
Optimization

Q1) i. The optimal value function of a linear program

$$\max c^T x, \quad Ax \leq b, \ x \geq 0$$

depends on the coefficients of the $m \times n$ matrix $A = (a_{ij})$, i.e., $f^o = f^o(a_{ij})$. Assume that Slater's condition holds and that optimal solutions of the primal and dual programs are unique. Derive a formula for the partial derivative $\frac{\partial f^o(a_{ij})}{\partial a_{ij}}$ at a given $a_{ij}$.

(ii) Consider the linear program

$$\max x_1 + 2x_2 + x_3,$$

$$x_1 + x_2 + x_3 \leq 4$$

$$x_1 + 2x_2 + 4x_3 \leq 7$$

$$x_1 + x_3 \leq 2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Check whether $x^* = (0, 3.5, 0)^T$ is an optimal solution. Then find $\frac{\partial f^o(a_{12})}{\partial a_{12}}$ at $a_{12} = 1$ and $\frac{\partial f^o(a_{22})}{\partial a_{22}}$ at $a_{22} = 2$.

Q2) Consider the convex program

$$\min x_1 + x_3,$$

$$x_1 + x_2 - x_3 = 2$$

$$x_1^2 + x_2^2 + x_3^2 \leq 4.$$

Using a first-order characterization of optimality, check whether the point $x^* = (0, 2, 0)^T$ is an optimal solution. If this point is not optimal, find a better feasible point using a numerical method of your choice.

Q3) This is a control navigation problem of Zermelo. The problem can be formulated in the Cartesian $(z_1, z_2)$ plane. Imagine a target, disc of radius $r = 1$, that is moving on the water parallel with the $z_1$-axis in its positive direction. The disc is moving with a constant speed $c = 2$ relative to the speed of the water, which is $V = 1$ also in the direction of positive $z_1$-axis. At $t = 0$, the coordinates of the center of the disc are $z_1 = -10, z_2 = 5$. A patrol boat at the origin is ready to intercept the target. It will travel with a constant speed $v = 3$ relative to the current. The problem is to find the boat's constant steering angle $\theta$ relative to the $z_1$ axis that would bring the boat to the target in shortest time.

i. Formulate the problem as a nonlinear program in $(\theta, t)$-variables.

ii. Sketch the feasible set in the $(\theta, t)$-plane, $0 \leq \theta \leq \pi$ and determine the optimal angle $\theta^*$ that solves the problem. Do not insist on numerical accuracy.
iii. Using an appropriate second order sufficient condition for optimality outline a procedure that can be used to verify that the numerical solution \((\theta^*, t^*)\), that you have found in (ii), is indeed an isolated local minimum. Can the boat reach the target in less time if it is allowed to change the steering angle during the interception?

Q4) Consider a convex hull \(C\) of finitely many points in \(\mathbb{R}^n\) and a continuous convex function \(f(x)\) defined on \(C\). Prove that \(f(x)\) assumes its global maximum on an extreme point of \(C\).
Q1) Consider the PDE

$$\frac{\partial}{\partial x} \left( e_x \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( e_y \frac{\partial u}{\partial y} \right) = 0, \quad x^2 + y^2 < 2,$$

with boundary condition $u(x, y) = x^2$ on $x^2 + y^2 = 2$. Write the weak form of the PDE. Show that the PDE has a unique weak solution (i.e., prove existence and uniqueness of solutions).

Q2) Let $u(x, y)$ be a harmonic function on the bounded domain $\Omega$ which lies in the first quadrant of $\mathbb{R}^2$ and is bounded by the two axes and the parabola $y = 4 - x^2$. Suppose also that along the boundary of $\Omega$, $u(0, y) = 0$, $u(x, 4-x^2) = 0$ and $u(x, 0) = x(4-x^2)$. Show that on $\Omega$, $0 < u(x, y) < x(4-x^2-y^2)$. State precisely any theorems you use.

Q3) (a) Use Fourier transforms to solve the following initial boundary value problem

$$\begin{cases} u_t(x, t) = u_{xx}(x, t) + \alpha u_x(x, t), & \text{for } x \in \mathbb{R}, \ t \geq 0, \\ u(x, 0) = u^0(x), & \text{for } x \in \mathbb{R} \end{cases}$$

where $u^0(x) \in L^2(\mathbb{R})$.

(b) Verify that $\|u(\cdot, t)\| \leq \|u^0\|$ for all $t \geq 0$ and that $\lim_{t \to \infty} \|u(\cdot, t)\| = 0$, where $\|u\|$ denotes the norm in $L^2(\mathbb{R})$.

Hint: $\int_{-\infty}^{\infty} e^{i\xi x} e^{-p\xi^2} \ d\xi = \sqrt{\frac{\pi}{p}} e^{-x^2/4p}$.

Q4) The initial value problem

$$\begin{cases} 2uu_x + u_t = 0 \\ u(x, 0) = \begin{cases} 0 & \text{if } \tau > 0 \\ \sqrt{\tau} & \text{if } \tau \leq 0. \end{cases} \end{cases}$$

has a shock along a curve of the form $x = Ct^2$. Find the solution and the precise location of the shock.
Numerical Analysis
Do any 3 out of 4 problems

Q1) (a) Define the relative condition number of a numerical problem. State what it means for an algorithm to be backward stable.

(b) Justify (with proof) the following result: A backward stable algorithm applied to a well-conditioned problem gives an accurate result.

(c) Consider the problem \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \) defined by \( f(x, y) = xy^T \), that is, the outer product. Equip \( \mathbb{R}^n \) with the 2-norm and \( \mathbb{R}^{n \times n} \) with the induced matrix 2-norm. What is the relative condition number of this problem with respect to perturbations in \( x \) as a function of nonzero \( x \) and \( y \)? Justify your answer.

(d) Explain why there cannot be a backward stable numerical algorithm for computing the outer product of two vectors.

Q2) (a) Suppose the matrix \( B \in \mathbb{R}^{n \times n} \) has Choleski decomposition \( B = C^T C \). Let \( w \in \mathbb{R}^n \). What is the Choleski decomposition of the matrix

\[
\begin{bmatrix}
1 & w^T \\
 w & B + ww^T
\end{bmatrix}
\]

(b) Let \( b \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times n} \), and is singular. Explain how you would construct a best solution, in the least squares sense, of the linear system \( Ax = b \). You may use exact arithmetic.

(c) Suppose \( R \in \mathbb{R}^{n \times n} \) is an upper triangular matrix with exactly \( k < n \) nonzero diagonal entries. What is the most that the rank of \( R \) could be? What is the least the rank of \( R \) could be? Justify your answer.

(d) Given any three distinct nodes in \( \mathbb{R}^2 \), is it always possible to interpolate a given function through them with a function of the form \( a + bx + cy \)? Justify your answer.

Q3) Consider Runge-Kutta method

\[
\begin{array}{c|ccc}
0 & 0 & 0 \\
2/3 & 1/3 & 1/3 \\
1/4 & 3/4 \\
\end{array}
\]

for approximating the solution of \( \dot{u} = f(u, t) \), \( u(0) = u_0 \). Write down the equations which define \( u_{n+1} \) in terms of \( u_n \) and describe an algorithm for finding \( u_{n+1} \).

Find the stability function \( R(z) \) of this method such that \( u_{n+1} = R(z)u_n \) when \( f(u, t) = \lambda u \).

Use the stability function to show that the method is not A-stable.

By expanding the stability function as a power series in \( z \) determine an upper bound for the order of the method. Hence by using the Runge-Kutta order conditions (which may be used without proof, but must be stated clearly), determine the order of the method.
Q4) Let \( u(x, t) \) be the solution of \( u_t = u_{xx} \), for \( t \geq 0, x \in [0, 1] \) with \( u(0, t) = \alpha, u(1, t) = \beta, \) and \( u(x, 0) = u_0(x) \).

Consider a numerical solution defined by the finite difference scheme

\[
\frac{u_{j}^{n+1} - u_{j}^{n}}{\Delta t} = \frac{u_{j+1}^{n+1} - 2u_{j}^{n+1} + u_{j-1}^{n}}{\Delta x^2}.
\]

Use Von Neumann stability analysis to show that this method is unconditionally stable. Show that the scheme satisfies a maximum principle,

\[
\max_j |u_{j}^{n+1}| \leq \max_j |u_{j}^{n}|, \quad \forall n \geq 0.
\]

Find the truncation error of the scheme, and show that if \( \mu = \Delta t/\Delta x^2 \) is kept constant the scheme is not a consistent approximation to \( u_t = u_{xx} \). Does the scheme define a consistent approximation to a different PDE? Can the scheme be made consistent by changing the relationship between \( \Delta t \) and \( \Delta x \)?

Finally, suppose that the boundary condition at \( x = 0 \) is replaced by \( u_x(0, t) = \alpha \). How can this boundary condition be discretized so as to obtain second order accuracy?