INSTRUCTIONS

Answer six of the nine questions.
Do not do things not asked for.
Quote the facts you use carefully.
Formulate your arguments with care,
and show all your calculations.
1. (a) Write down the definition of the Fourier transform and its inverse. Applying this to an odd function derive the formulae for the Fourier sine transform and its inverse.

(b) Solve the following initial boundary value problem (in which $k$ is a positive constant):

$$\frac{\partial u}{\partial x}(x, t) = k \frac{\partial^2 u}{\partial x^2}(x, t), \text{ for } x \geq 0, \ t \geq 0;$$

$$u(0, t) = h(t), \text{ for } t \geq 0;$$

$$u(x, t) \text{ and } u_x(x, t) \to 0 \text{ as } x \to \infty;$$

$$u(x, 0) = 0, \text{ for } x \geq 0.$$

(Note: $\int_0^\infty e^{-\alpha y^2} \cos \omega y \, dy = \frac{1}{2} \sqrt{\frac{\pi \omega}{\alpha}} e^{-\omega^2/4\alpha}$).

2. Let $T(x, t)$ denote the ground temperature at depth $x$ below the surface, at time $t$. Suppose that the surface temperature varies with time in the form:

$$T(0, t) = T_0 + A_0 \cdot \cos \omega t \quad (t > 0) \quad (1)$$

where $T_0$ is the average temperature, $A_0$ is the amplitude of temperature fluctuation, and $\omega$ is the frequency ($T_0, A_0, \omega$ are known). Suppose that the temperature distribution underground is subject to the heat conduction equation:

$$\frac{\partial T}{\partial t} = \kappa T \frac{\partial^2 T}{\partial X^2} \quad (2)$$

and that, as $x \to \infty$

$$T \to T_\infty \quad (3)$$

where $T_\infty$ is a constant to be determined.

Find

(a) the underground temperature distribution $T(x, t),$

(b) $T_\infty,$ the temperature in the deep underground,

(c) the depth $x_\ast,$ where the temperature fluctuation is reduced to 1%.
3. Consider an infinite non-homogeneous string, which is composed of two different materials joined at $x = 0$. The displacements $u_1$ and $u_2$ of the string segments are subject to the wave equations:

\[
\begin{align*}
\left( \frac{\partial^2}{\partial t^2} - c_1^2 \frac{\partial^2}{\partial x^2} \right) u_1(x, t) &= 0, \quad x < 0 \\
\left( \frac{\partial^2}{\partial t^2} - c_2^2 \frac{\partial^2}{\partial x^2} \right) u_2(x, t) &= 0, \quad x > 0
\end{align*}
\]

Suppose that starting from $t = 0$, there is an incoming right running wave

\[
u_1(x, t) = \begin{cases} 
\Phi \left( t - \frac{x}{c_1} \right) & x < c_1 t \\
0 & x > c_1 t
\end{cases}
\]

This incoming wave will be reflected at $x = 0$ and also transmitted into the region ($x > 0$).

Determine

(a) the transmission wave and the reflection wave,

(b) the transmission rate $T$ and reflection rate $R$ of the wave energy, supposing that the wave energy is measured by its magnitude.
1. Gauss Quadrature and Orthogonal Polynomials

(a) Show for \( n = 1 \) and \( n = 2 \) that by varying both the nodes \( x_i \) and weights \( w_i \) in

\[
\int_{-1}^{1} f(x) dx \approx \sum_{i=1}^{n} w_i f(x_i)
\]

you can get an exact quadrature formula for polynomials of degree \( 2n - 1 \). Derive those formulas for \( n = 1, 2 \).

(b) What makes it difficult to continue this process for \( n > 2 \)?

(c) How would you construct orthogonal polynomials on the interval \([−1, 1]\)? Give a precise procedure and use it to compute the first two orthogonal polynomials. How are those polynomials called?

(d) Show that orthogonal polynomials satisfy a 3 term recurrence relation

\[
x p_{k-1} = \beta_{k-1} p_{k-2} + \alpha_k p_{k-1} + \beta_k p_k.
\]

(e) Show how you can use these orthogonal polynomials to construct the desired quadrature rules described in the first part. What are the nodes \( x_i \) and the weights \( w_i \)?

(f) Write the recurrence relation simultaneously to obtain a matrix formulation. Show how you can extract from this matrix both the nodes \( x_i \) and the weights \( w_i \). (This part is challenging.)

2. Eigenvalue computations

(a) Derive the power method to compute the largest eigenvalue \( \lambda_1 \) of a given symmetric matrix \( A \).

(b) Assuming that \( \lambda_1 > \lambda_2 \geq \ldots \) prove that your method converges.

(c) Suppose your starting vector has no component in the direction of the first eigenvector associated with \( \lambda_1 \). What would happen mathematically? What happens numerically and why?

(d) What can you do to find the smallest eigenvalue? Describe how you would implement this algorithm.

(e) Suppose you need the two largest eigenvalues. Derive a method similar to the power method to find the two values simultaneously. What condition do you need on the eigenvalues for the method to converge?

(f) Derive the QR algorithm as a generalization of this process.

(See parts (g) and (h) next page.)
(g) What is the order of the computational cost per step if this algorithm is implemented
naively?

(h) Derive an initial transformation $A = QHQ^T$ such that each step of the QR algorithm
becomes an order of magnitude faster.

3. Finite differences for the heat equation $u_t = u_{xx}$

(a) Apply leap frog to the heat equation. Is the scheme explicit or implicit?

(b) Give a definition of the truncation error. What is the truncation error of leap frog?

(c) Analyze the stability of this scheme.

(d) Du Fort-Frankel improved leap frog by taking an average in time for the middle term
in the space discretization. Write down the new scheme. Is this explicit or implicit?

(e) What is the truncation error of this scheme? For what choices of $h$ and $\Delta t$ does the
scheme converge?

(f) Analyze the stability of this scheme. Can you choose $h$ and $\Delta t$ as you wish?
OPTIMIZATION