INSTRUCTIONS

Attempt all eight questions.
Do not do things not asked for.
Quote the facts you use carefully.
Formulate your arguments with care,
and show all your calculations.
1. (a) Let \( g : [0, 1] \to \mathbb{R} \). What does it mean for \( g \) to be continuous?

(b) State the Bolzano-Weierstrass Theorem.

(c) If \( g : [0, 1] \to [0, \infty] \) is a continuous function show that \( g \) attains its infimum.

(d) Let \( f : [0, 1] \to [0, 1] \) be such that \( |f(x) - f(x')| < |x - x'| \) for all \( x, x' \in [0, 1] \) with \( x \neq x' \). Show that \( f \) has a unique fixed point. \textit{Hint:} Show that the function \( g(x) = |x - f(x)| \) is continuous. Where does \( g \) take its infimum?

2. For each of the following sequences of functions \( (f_n)_{n=1}^\infty \) defined on \( ]0, \infty[ \), determine the pointwise limit. Determine also if convergence is uniform

(\( \alpha \)) on the whole of \( ]0, \infty[ \);

(\( \beta \)) on every interval \([a, b]\) with \( 0 < a < b < \infty \).

(i) \( f_n(x) = \frac{nx}{n[x]} \), (ii) \( f_n(x) = \frac{nx}{1 + n^2x^2} \), (iii) \( f_n(x) = xe^{-nx} \),

(iv) \( f_n(x) = \left( \frac{\sin(nx)}{n\sin(x)} \right)^2 \) for \( x \) not an integer multiple of \( \pi \), \( f_n \) extended continuously to \( ]0, \infty[ \).

For any real number \( t \), we have denoted above \( \lceil t \rceil = k \), where \( k \) is the unique integer such that \( k - 1 < t \leq k \).

3. Let \( V \) be a finite dimensional vector space over \( \mathbb{C} \), the field of complex numbers. Let \( T : V \to V \) be a non-singular linear operator, \( k \) a positive integer. Prove that if \( T^k \) is diagonalizable, then \( T \) is diagonalizable. Show that “\( T \) is non-singular” cannot be omitted.

4. (a) Prove that the square matrix \( A \) with entries in a field is non-singular (meaning \( AX = 0 \) implies \( X = 0 \)) if and only if \( \det(A) \neq 0 \). [This is standard material; algebraic properties of determinants may be freely used.]

(b) Let \( A = (a_{ij})_{m \times n} \) be a not-necessarily square matrix with entries in a field. An \( \ell \times \ell \)-subdeterminant of \( A \) is the determinant of any matrix

\[
\begin{vmatrix}
    a_{i_1,j_1} & \cdots & a_{i_1,j_\ell} \\
    \vdots & \ddots & \vdots \\
    a_{i_\ell,j_1} & \cdots & a_{i_\ell,j_\ell}
\end{vmatrix}
\]

composed of entries in \( \ell \) rows and columns of the given \( A \), for any \( \ell \leq \min(m, n) \). Prove that the rank of \( A \) (defined as the dimension of the column space of \( A \)) equals the largest \( \ell \) for which there is a non-zero \( \ell \times \ell \) subdeterminant of \( A \).
5. Suppose that $X$ and $Y$ are independent random variables.

(a) Let $f(x, y)$ be non-negative, and let $g(x) = Ef(x, Y)$. Show that $Ef(X, Y) = Eg(X)$.

(b) Suppose that $E|X + Y| < \infty$. Show that $E|Y| < \infty$.

6. (a) Suppose that $\{Z_n, n \geq 1\}$ is a sequence of non-negative r.v.’s on a probability space $(\Omega, \mathcal{F}, P)$, and that $Z_n \leq Z$, where $EZ < \infty$. Show that if $Z_n \xrightarrow{P} 0$, then $EZ_n \rightarrow 0$.

(b) Suppose that $\{X_n, n \geq 1\}$ is a sequence of r.v.’s on a probability space $(\Omega, \mathcal{F}, P)$, and that $|X_n| \leq Y$, where $E|Y| < \infty$. Show that if $X_n \xrightarrow{P} X$, then $E|X| < \infty$ and $EX_n \rightarrow EX$.

7. (a) State both Borel-Cantelli lemmas. Prove one of them.

(b) Suppose that $X_1, X_2, \ldots$ are independent r.v.’s with $P\{X_n = 1\} = p_n$ and $P\{X_n = 0\} = 1 - p_n$. Show that

i. $X_n \xrightarrow{P} 0$ iff $p_n \rightarrow 0$.

ii. $X_n \xrightarrow{a.s.} 0$ iff $\sum_{n=1}^{\infty} p_n < \infty$.

8. Suppose $X$ and $Y$ are r.v.’s on $(\Omega, \mathcal{F}, P)$.

(a) Suppose $X$ and $Y$ have joint density given by

$$f(x, y) = \begin{cases} 2 & \text{if } 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $E(X|Y)$.

(b) Suppose $X$ and $Y$ are independent, and that $\phi : \mathbb{R}^2 \rightarrow [0, \infty)$ is measurable. For each $x$, let $h(x) = E\phi(x, Y)$. Show that $h(X)$ is a version of $E[\phi(X,Y)|X]$. 

PROBABILITY