McGill University
Department of Mathematics and Statistics
Part A Examination
Statistics α

Date: Monday, May 8, 1995
Time: 13:00 - 17:00

Instructions:

1. All questions on both Papers α and β are compulsory.

2. Satisfactory performance on questions 1-4 on Paper α is necessary in addition to an overall pass.

Questions:

1. Let $T$ be a linear operator on a finite-dimensional complex inner product space. We define $T^*$ by $<Tv,w> = <v,T^*w>$ for all $v, w \in V$. Assume that $T$ is normal, which means that $T$ commutes with $T^*$.

(a) Show that $Tv = 0$ if and only if $T^*v = 0$.

(b) Show that if $v$ is an eigenvector of $T$, then it is also an eigenvector of $T^*$.

(c) Show that if $v$ is an eigenvector for $T$, then $v^\perp = \{w \in V | <v,w> = 0\}$ is a $T$-invariant subspace of $V$ and that $T$ induces a normal operator on $v^\perp$.

(d) Show that $T$ is diagonalizable.

2. Let $W$ be a subspace of the finite-dimensional complex vector space $V$. Prove, by means of an explicit and natural isomorphism, that

$$(V/W)^* \simeq W^\perp.$$ 

Notation: $V/W$ is the quotient space.

$X^* = \{f : X \to \mathbb{C}, \text{ such that } f \text{ is a linear functional}\}$

$= \text{the (algebraic) dual of } X.$

$W^\perp = \{f \in V^*, \text{ such that } f(w) = 0 \ \forall w \in W\}.$
3. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 
1 & \text{if } \frac{1}{2n+1} \leq x \leq \frac{1}{2n}, \ n = 1, 2, \ldots \\
0 & \text{otherwise}.
\end{cases}$$

Prove that $f$ is Riemann integrable and that $\int_0^1 f(x) \, dx = 1 - \log 2$.

4. (a) Define the concept $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly for $x \in A$.

(b) STATE the Weierstrass $M$-test concerning the uniform convergence of a series of functions.

(c) Prove that $\sum_{n=1}^{\infty} (x \log x)^n$ converges uniformly for $x \in (0, 1]$.

5. Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space. A set $E \in \mathcal{F}$ is an atom of $\mu$ if $\mu(E) > 0$ and $\mu(F) = 0$ or $\mu(E)$ whenever $F \in \mathcal{F}$ and $F \subset E$.

Show that there can only be countably many atoms, provided we identify atoms $A$ and $B$ for which $\mu(A \Delta B) = 0$.

6. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $EX_1 = 0$. Let $Y_n = X_n/n$. Show that, if

$$E[|X_1| \log(1 + |X_1|)] < \infty, \quad (*)$$

then $\sum_{k=1}^{n} Y_k$ converges to a finite limit with probability one, and construct an example to show that this need not be true if we drop the condition $(*)$.

7. Suppose $X_i$ are i.i.d with $EX_i = 0$, $EX_i^2 = 1$. Let $Z_n = \sum_{i=0}^{n} X_i I_{\{X_{n_i} > 0\}}$. Find the limiting distribution of $Z_n/\sqrt{n}$. 

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8. (a) State the Strong Law of Large Numbers

(b) Let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed random variables with \( E(X_i^-) < +\infty \) and \( E(X_i^+) = +\infty \). Show that if \( X_n = \sum_{i=1}^n X_i \), then \( \frac{S_n}{n} \to +\infty \) almost everywhere. (Hint: For fixed \( c \geq 0 \), let \( X_i^+ = X_i^+ I_{\{X_i^+ > c\}} + X_i^+ I_{\{X_i^+ \leq c\}} \).

(c) Show that for any sequence of random variables, \( \{X_n\} \), \( \frac{S_n}{n} \to p \) 0 implies \( \frac{X_n}{n} \to 0 \).