McGill University
Department of Mathematics and Statistics
Part A Examination
Pure and Applied Mathematics $\alpha$

Date: Monday, May 11, 1998
Time: 9:00 A.M. - 1:00 P.M.
Room: BURN 1120

Instructions

1. All questions are compulsory
2. Satisfactory performance on questions 1 - 4 on Paper $\alpha$ is necessary in addition to an overall pass.

1. Prove or disprove by counterexample each of the following statements about a sequence of positive real numbers:

   (a) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

   (b) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} na_n^2$ is convergent.

   (c) If $\frac{a_{n+1}}{a_n} \leq 1 - \frac{2}{n}$ for $n > 10$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

2. (a) What does it mean for a real-valued function $f$ to be Riemann integrable on $[0, 1]$?
(b) Show that every real-valued continuous function on $[0, 1]$ is Riemann integrable on $[0, 1]$.
(c) Let $f$ be a real-valued continuous function on $[0, 1]$ such that

$$\int_{0}^{1} f(x)h(x)dx = 0$$

for all real-valued continuous functions $h$ on $[0, 1]$. Show that $f$ is identically zero.

3. (a) Define the terms removable singularity, pole, essential singularity.
(b) Let $f(z) = \exp\left(\frac{a}{z} + \frac{b}{z^2}\right)$ for $z \in \mathbb{C} \setminus \{0\}$, where $a, b \in \mathbb{C}$ and $b \neq 0$. What type of singularity does $f(z)$ have at the point $z = 0$? Prove your assertion.
4. (a) State the Residue Theorem.
(b) Using the Residue Theorem determine the integral
\[
\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1} \, dx,
\]
justifying your answer in detail.

5. Use the Divergence Theorem to evaluate
\[
\int_{\partial \Omega} (xy, yz, zx) \cdot \tilde{n} \, d\sigma
\]
where \( \Omega \) is the region given by \( x^2 + y^2 \leq z^2, 0 \leq z \leq 1 \) and where \( \tilde{n} \) and \( \sigma \) are respectively the outward normal and surface area measure on \( \partial \Omega \).

6. Let \( A, B \) be \( n \times n \) matrices and suppose that \( A, B \) are diagonalizable. Show that there is an invertible matrix \( P \) such that \( P^{-1}AP \) and \( P^{-1}BP \) are both diagonal matrices if and only if \( AB = BA \).

7. If \( A, B \) are respectively \( m \times n \) and \( n \times p \) matrices, show that the dimension of the null space of \( AB \) is at most the sum of the dimensions of the null spaces of \( A \) and \( B \). When do we have equality?

8. (a) State an existence and uniqueness theorem for the initial value problem of the form
\[
\frac{dx}{dt} = f(x, t); \quad x(t_0) = x_0.
\]
(b) Consider the initial value problem
\[
t^2 \frac{dx}{dt} = x; \quad x(t_0) = x_0
\]
for \( t_0 \) in an open interval \( I \).
Show that
i. there is a unique solution if \( t_0 \neq 0 \) and find the maximal interval on which the solution is valid;
ii. there is no solution for \( t_0 = 0, \ x_0 \neq 0 \);
iii. there are infinitely many solutions for \( t_0 = 0 \) and \( x_0 = 0 \), and these solutions can each be extended to all of \( \mathbb{R} \).