1. All questions are compulsory.
2. Satisfactory performance on questions 1 – 4 on Paper α is necessary in addition to an overall pass.

1. (a) Show $T$ is a normal operator (commutes with its adjoint) on complex inner product space $V$ if and only if $T = T_1 + iT_2$ where $T_1$ and $T_2$ are commuting self adjoint operators on $V$.

(b) Given $A = (a_{jk})$ with $a_{jk} = \frac{1}{j+k+1}$, $0 \leq j, k \leq n$. Show that $A$ defines a positive definite quadratic form. [Hint: Relate this to an inner product on the space of polynomials of degree at most $n$.]

(c) Give an example of an operator on an inner product space which does not have an adjoint.

2. Find a basis of $\mathbb{R}^3$ which puts the matrix $A = \begin{pmatrix} 0 & 0 & 12 \\ 1 & 0 & -16 \\ 0 & 1 & 7 \end{pmatrix}$ into Jordan canonical form (Note: $+3$ is an eigenvalue.)

3. (a) Prove that for any positive integer $k$,

$$
\lim_{n \to \infty} \frac{1}{2^n} \binom{n}{k} = 0.
$$

(b) If $(a_n)$ is a sequence of real numbers tending to a limit $a$, prove that

$$
\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{n} \binom{n}{k} a_k = a.
$$
4. (a) Let \( (f_n) \) be a sequence of (real-valued) functions defined on \([a, b]\) which converges uniformly on \([a, b]\) to \(f\). If each \( f_n (n \in \mathbb{N}) \) is continuous on \([a, b]\), prove that \(f\) is also continuous on \([a, b]\).

(b) For \(x \in [0, 1]\) discuss the uniformity of convergence of the sequences given by
   
   \[
   \begin{align*}
   (i) \quad & \frac{nx}{1 - nx + 2n^2 x^2}, \\
   (ii) \quad & 2n^2 xe^{-n^2 x^2}.
   \end{align*}
   \]

   In each case compare \(\lim_{n \to \infty} \int_0^1 f_n dx\) and \(\int_0^1 f dx\).

5. Let \(\vec{F} = (x^2 + 2x + y)\vec{i} + (y^2 + 2y + z)\vec{j} + (2z + x)\vec{k}\).

   Let \(D\) be the region which is contained between the two surfaces \(S_1 : 2x^2 + 3y^2 = z\) and \(S_2 : 9 - (x^2 + 2y^2) = z\). Evaluate \(\iint_D \vec{F} \cdot \vec{n} \, ds\) where \(\partial D\) is the boundary of \(D\) oriented with the normal pointing into \(D\). State the theorem of vector calculus you may use to evaluate the above integral.

6. Evaluate
   \[
   \int_0^\infty \frac{\sin^2 2x}{x^2(x^2 + 1)} \, dx.
   \]

7. Consider an entire function \(f(z)\) for which:

   \[
   \Im f'(z) = -e^x \sin x, \quad f(0) = 5 + 3i, \quad f(\pi/2) = 2i.
   \]

   Express \(f(z)\) as a function of \(z\).

8. Solve \(xy' + y = (xy)^{3/2}\), \(y(1) = 9\). What is the maximal interval of existence of that solution?