INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.
Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

1. Suppose that $T_1$ and $T_2$ are linear operators on the space $V$ (over the field $F$) and that $p_1(X)$ and $p_2(X)$ are polynomials with coefficients in $F$. Suppose also that $T_1T_2 = T_2T_1$. Show that each of the spaces $\ker(p_2(T_2))$ — the kernel of $p_2(T_2)$ — and $\text{im}(p_2(T_2))$ — the image of $p_2(T_2)$ — is invariant under the operator $p_1(T_1)$.

2. (a) Suppose that $A$ is an $n \times n$ real matrix with $n$ distinct real eigenvalues. Show that $A$ can be written in the form $\sum_{j=1}^{n} \lambda_j I_j$ where each $\lambda_j$ is a real number and the $I_j$ are $n \times n$ real matrices with $\sum_{j=1}^{n} I_j = I$, and $I_j I_l = 0$ if $j \neq l$.

(b) Give a $2 \times 2$ real matrix $A$ for which such a decomposition is not possible and justify your answer.

3. Show that, if $a$ is a positive real number, and $f$ is a continuous real-valued function on $[0, a]$, then

$$\left( \int_0^a e^x f(x)^2 dx \right) \cdot \left( \frac{e^{3a} - 1}{3} \right) \geq \left( \int_0^a e^{2x} f(x) dx \right)^2.$$ 

Identify those functions $f$ such that equality holds.

4. Let $A$ be a $10 \times 10$ matrix with real entries, and suppose that its eigenvalues are 4 and 7. You are given the information that $\text{null}(A - 4I)$ has dimension 3, $\text{null}(A - 4I)^2$ has dimension 5, $\text{null}(A - 4I)^3$ has dimension 7, $\text{null}(A - 4I)^4 = \text{null}(A - 4I)^5$ has dimension 8, $\text{null}(A - 7I)$ has dimension 1 and $\text{null}(A - 7I)^2 = \text{null}(A - 7I)^3$ has dimension 2. Find the Jordan canonical form $J$ of $A$. Also indicate how to find the columns of a matrix $P$ such that $P^{-1}AP = J$. [$\text{null}(B)$ is the null space of $B$, the set of vectors $\vec{v}$ such that $B\vec{v} = \vec{0}$.]
Solve any three out of the four questions 5, 6, 7, 8.

5. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a bounded function. For a given (finite) partition \( P \) of \([a, b]\) denote by \( I_k, \ k = 1, 2, \ldots, n \) the intervals of the partition, denote \( M_k := M_k(P) := \sup_{x \in I_k} f(x) \), \( m_k := m_k(P) := \inf_{x \in I_k} f(x) \), and let \( |I_k| \) denote the length of \( I_k \).

(a) State the Riemann Criterion for Riemann integrability of \( f \).

(b) Prove that the Riemann integrability of \( f \) is equivalent to the following property: For any \( \epsilon_1, \epsilon_2 > 0 \) there is a partition \( P \) of \([a, b]\) such that if \( A := \{ k \mid M_k(P) - m_k(P) > \epsilon_1 \} \) then

\[
\sum_{k \in A} |I_k| \leq \epsilon_2
\]

6. Let \( f \geq 0 \) be a continuous function on \([0, 1]\). Suppose there exists \( C \in \mathbb{R} \) such that for all \( x \in [0, 1] \),

\[
f(x) \leq C \int_0^x f(t) \, dt
\]

Prove that \( f = 0 \).

7. Let \( f \) be a continuous function on \([0, \infty)\) with \( \lim_{x \to \infty} f(x) = 0 \).

(a) If \( f \geq 0 \), prove that there exists \( c \in [0, \infty) \) such that \( f(x) \leq f(c) \) for all \( x \geq 0 \).

(b) If the hypothesis \( f \geq 0 \) is omitted in (a), show that the statement is false by giving a counterexample.

8. Suppose that \( \{a_n\} \) is a real sequence with \( \lim_{n \to \infty} a_n = 0 \) and that \( \{b_n\} \) is a nonnegative sequence with

\[
\lim_{n \to \infty} (b_1 + b_2 + \cdots + b_n) = \infty.
\]

Prove that

\[
\lim_{n \to \infty} \left( \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{b_1 + b_2 + \cdots + b_n} \right) = 0.
\]
Solve any three out of the four questions 9,10,11,12.

9. Consider the second order differential equation for \( u(x) \)
\[
\frac{d^2u}{dx^2} + \lambda u = 1, \quad -\infty < x < \infty,
\]
where \( \lambda \) is a real number.

(a) Give two-parameter families of real solutions to this equation for each of the three cases \( \lambda < 0, \lambda = 0, \lambda > 0 \).

(b) Consider the boundary value problem
\[
\frac{d^2u}{dx^2} + \lambda u = 1, \quad 0 \leq x < \infty, \quad u(0) = 0.
\]
For what values of \( \lambda \) are there solutions \( u(x) \) that are bounded for all \( x > 0 \)? Justify your answer.

10. Consider the ellipse in 3-space given by the intersection of the cylinder \( x^2 + y^2 = 1 \) and the plane \( 2x + y - z = 4 \). Find the point with the largest \( z \)-value on the ellipse. Hint: Use Lagrange multipliers.

11. Let \( \mathbf{F} = 3xz\mathbf{i} - x\mathbf{j} - y\mathbf{k} \). Let \( S \) denote the portion of the cylinder \( y^2 + z^2 = 1 \) that satisfies \( x, y, z > 0 \) and \( x < 1 \). Find the flux of \( \mathbf{F} \) outward across \( S \).

12. Let \( f(z) = g(z)/h(z) \), where \( g \) and \( h \) are both analytic at \( z_0 \). If \( g(z_0) \neq 0 \) and \( z_0 \) is a zero of order 2 of \( h \), prove that the residue of \( f \) at \( z_0 \) is
\[
\frac{2g'(z_0)}{h''(z_0)} - \frac{2g(z_0)h'''(z_0)}{3[h''(z_0)]^2}.
\]