INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.
1. Suppose that $T_1$ and $T_2$ are linear operators on the real vector space $V$. Assume that $T_1$ is one-to-one, but that $\vec{v}$ is a nonzero vector such that $T_2\vec{v} = 0$; also suppose that $T_2T_1 - T_1T_2 = T_1$.

(a) Show that for each non-negative integer $n$ the vector $T_1^n\vec{v}$ is an eigenvector of $T_2$ and find the corresponding eigenvalue.

(b) Show that $V$ is infinite-dimensional.

2. Suppose that $A$ is an $m \times n$ matrix with $(i, j)$-entry $a_{i,j}$. Whenever $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, we call
\[
\det \begin{pmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_k} \\
\vdots & \vdots & \vdots \\
a_{i_k,j_1} & \cdots & a_{i_k,j_k} \end{pmatrix}
\]
a $k \times k$ subdeterminant of $A$.

Show that the rank of $A$ is the biggest number $k$ such that there is a nonzero $k \times k$ subdeterminant of $A$.

3. (a) Suppose that $F$ is an algebraically closed field of characteristic not equal to 2, and $A$ is a square matrix over $F$ such that $A^2$ is diagonalizable and $A$ is invertible, show that $A$ is also diagonalizable.

(b) Give an example of an invertible, non-diagonalizable matrix $A$ over an algebraically closed field of characteristic 2 such that $A^2$ is diagonalizable.

(c) Give an example of a non-invertible matrix $A$ over an algebraically closed field of characteristic different from 2 such that $A$ is not diagonalizable but $A^2$ is.

4. If $k$ and $n$ are positive integers with $k \leq n$ and $F$ is a field, let $X_k$ denote the set of all $k$-dimensional subspaces of $F^n$. Prove that $GL_n(F)$ acts transitively on $X_k$. 

2
Single variable real analysis

Solve any three out of the four questions 5,6,7,8.

5. (a) Let \( f : \mathbb{R} \to \mathbb{R} \) be uniformly continuous on the whole real line and for \( x \in \mathbb{R} \) define

\[
 f_n(x) = f(x + n^{-1}); \quad n = 1, 2, 3, \ldots
\]

Show that the sequence \((f_n)\) converges uniformly on \( \mathbb{R} \) to \( f \).

(b) Determine whether the following functions are uniformly continuous on the open interval \((0, \infty)\) and justify your assertions:

\[
(i) \quad f(x) = x \cos\left(\frac{1}{x}\right), \quad (ii) \quad g(x) = \frac{x^3}{x + 1}.
\]

6. (a) If \( x \in (-\infty, \infty) \), prove that \( e^x \geq 1 + x \).

(b) If \( 0 \leq x \leq 1 \), prove that

\[
\frac{x}{5} + e^{(-x/4)} \leq 1.
\]

7. Let \( f(x) \) be a continuous positive function on \( \mathbb{R} \) and let

\[
\varphi(x) = \frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt}, \quad x > 0.
\]

(a) Prove that \( \varphi(x) \) is an increasing function.

(b) Suppose that \( \lim_{x \to \infty} f(x) = 1 \). Find

\[
\lim_{x \to \infty} \frac{\varphi(x)}{x}.
\]

8. Consider the power series

\[
\sum_{n=1}^{\infty} \frac{10^{\nu(n)}}{n} (1 - x)^n,
\]

where \( \nu(n) \) is the number of digits of the number \( n \) in base 10. Find all real numbers \( x \) for which this power series converges.
Solve any three out of the four questions 9,10,11,12.

9. (a) Prove that
\[ \left| \frac{z - w}{1 - \overline{w}z} \right| = 1 \]
whenever \( z, w \) are complex numbers with \( |z| = 1 \) and \( w \neq z \).

(b) Let \( P \) be a nonzero polynomial over \( \mathbb{C} \) such that \( P(0) = 1 \). Prove that there is a polynomial \( Q \) of the same degree as \( P \) such that \( Q(0) = 1 \), \( Q \) has no zeros in the open unit disc \( |z| < 1 \), and \( |Q(z)| \leq |P(z)| \) whenever \( |z| = 1 \).

10. Consider the ODE
\[ y''' + 4y' = e^{-t}. \]

(a) Find the general solution.

(b) Find a solution which satisfies \( \lim_{t \to \infty} y(t) = 1 \).

11. Let \( u \) be a \( C^2 \) function on the closed interval \([a, b]\) such that \( u'' = u \). For any \( C^2 \) function \( y \) on \([a, b]\) satisfying the boundary conditions \( y(a) = u(a) \), \( y(b) = u(b) \), define
\[ J(y) = \int_a^b (y'(x))^2 + y(x)^2 \, dx. \]
For all such \( y \), prove that \( J(y) \geq J(u) \). (Hint: Given \( y \), write \( y = u + \psi \).)

12. Evaluate
\[ \int_C \frac{\zeta \, d\zeta}{\zeta^2 - z^2}, \]
where \( C \) is the unit circle oriented counter-clockwise in the complex plane and \( z \in \mathbb{C}\setminus C \).