McGill University
Department of Mathematics & Statistics
Part A Examination
Pure & Applied Mathematics $\alpha$

Date: Monday, May 17, 1999
Time: 1:00 pm - 5:00 pm
Room: BURN 1234

INSTRUCTIONS

All questions are compulsory.

LINEAR ALGEBRA

1. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & -16 & 7 \end{pmatrix}.$$  

(a) Find polynomials $a(t)$ and $b(t)$ so that $a(t)(t - 2)^2 + b(t)(t - 3) = 1$.
(b) Find the projection matrices $a(A)(A - 2I)^2$ and $b(A)(A - 3I)$.
(c) Find a matrix $P$ so that $P^{-1}AP$ is in Jordan form.

2. Suppose that $T : V \to V$ is a Hermitian operator on a finite dimensional inner product space over the complex numbers.

(a) Prove that the eigenvalues of $T$ are real.
(b) Prove that $V$ has a basis consisting entirely of eigenvectors of $T$.

REAL ANALYSIS

1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $0 < a_{n+1} < a_n$ for $n = 1, 2, \ldots$ and such that $\sum_{n=1}^{\infty} a_n$ converges. Which of the following statements is necessarily true? For each assertion either give a proof or disprove by counterexample.

(i) $\sum_{n=1}^{\infty} na_n^2$ converges.
(ii) $\sum_{n=1}^{\infty} \frac{a_n a_{n+1}}{a_n - a_{n+1}}$ converges.
(iii) $\sum_{n=1}^{\infty} (-1)^{n-1} \sqrt{a_n}$ converges.
2. (a) State, but do not prove, Rolle's Theorem.
    (b) State and prove the Mean Value Theorem.
    (c) Let $\varphi$ be a continuous real-valued function on $[0, 1]$ such that $\varphi'(t)$ exists at each point of $(0, 1)$ and such that $\varphi'$ is increasing on $(0, 1)$. Establish the inequality
    \[\varphi(t) \leq (1 - t)\varphi(0) + t\varphi(1)\]
    for $0 \leq t \leq 1$.

COMPLEX ANALYSIS

State the residue theorem.

By integrating $\frac{e^{it}}{z^2 + a^2}$ around a suitable closed path, find $\int_0^\infty \frac{\cos x}{x^2 + a^2}$ where $a \in \mathbb{R}$, $a \neq 0$.

DIFFERENTIAL GEOMETRY

Let $\vec{\alpha}(s)$ be a unit speed curve on a sphere of radius $r$. Show that the curvature $K(s)$ of $\vec{\alpha}(s)$ satisfies $K(s) \geq \frac{1}{r}$, with equality for all $s$ if and only if the image of $\vec{\alpha}$ is contained in a great circle.

DIFFERENTIAL EQUATIONS

Find the general solution of

\[y'' - 5y' - 6y = 3xe^{-x} + 4\cos 2x.\]

CALCULUS

State Green's Theorem.

Evaluate

\[\int_{\partial G} [(\tan^2(x^3) - 2y^3) dx + (e^{(y^2+1)} + 2x^3) dy]\]

where $G$ is the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$. 