INSTRUCTIONS:

(i) There are 12 problems. Solve three of 1,2,3,4; three of 5,6,7,8; and three of 9,10,11,12.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.
Linear Algebra

Solve any three out of the four questions 1, 2, 3, 4.

**Problem 1**
Let $A$ be a linear operator. Prove that

$$\dim \text{Ker} \ A^{n+1} = \dim \text{Ker} \ A + \sum_{k=1}^{n} \dim(\text{Im} \ A^k \cap \text{Ker} \ A)$$

and

$$\dim \text{Im} \ A = \dim \text{Im} \ A^{n+1} + \sum_{k=1}^{n} \dim(\text{Im} \ A^k \cap \text{Ker} \ A)$$

**Problem 2**
Let $A$ be an invertible matrix.
Prove that if $\text{rank} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{rank} \ A$, then $D = CA^{-1}B$.

**Problem 3**
Let $E$ be a vector space over a field $k$ of dimension $n$. Let $T : E \to E$ be a linear map such that $T$ is nilpotent, that is $T^m = 0$ for some positive integer $m$. Show that there exists a basis $E$ over $k$ such that the matrix $A = (a_{ij})$ of $T$ with respect to this basis is strictly upper triangular, that is $a_{ij} = 0$ if $i \geq j$.

**Problem 4**
Let $F$ be a finite field with $q$ elements. Show that the order of $GL_n(F)$ is

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) = q^{n(n-1)/2} \prod_{i=1}^{n}(q^i - 1).$$
Single variable real analysis

Solve any three out of the four questions 5, 6, 7, 8.

**Question 5.** Define the function $f$ as follows:

$$f(x) = \begin{cases} 
0, & x \notin \mathbb{Q} \\
1/n, & 0 \neq x = (m/n) \in \mathbb{Q}, \ GCD(m, n) = 1 \\
1, & x = 0.
\end{cases}$$

Show that $f(x)$ is continuous at every $x \notin \mathbb{Q}$, and that $f(x)$ is not continuous at every $x \in \mathbb{Q}$.

**Question 6.** Let

$$f_n(x) = \frac{x}{1 + nx^2}.$$

i) Show that $f_n$ converges uniformly to a function $f$ on the interval $[-1, 1]$.

ii) Show that the equality $f'(x) = \lim_{n \to \infty} f'_n(x)$ holds for all $x \neq 0$, and fails for $x = 0$.

**Question 7.**

i) State Taylor’s Theorem with Lagrange’s form of the remainder term.

ii) Show that $\left| \ln(1 + x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n}\right) \right| < \frac{x^{n+1}}{n+1}$ in the case $x > 0$.

iii) Show that $\left| \ln(1 + x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n}\right) \right| < \frac{1}{n+1} \left| \frac{x}{1+x} \right|^{n+1}$ in the case $-1 < x < 0$.

**Question 8.** Find the radius of convergence of the following power series:

i) $\sum_n \frac{2^n}{n!} z^n$.

ii) $\sum_n \frac{4^n}{n^2} z^n$.

iii) $\sum_n \frac{n^3}{3^n} z^n$. 
Solve any three out of the four questions 9,10,11,12.

**Question 9.** Find a general solution of the equation

\[ y^{(iv)} - 5y'' + 4y = 80e^{3x}. \]

**Question 10.** The point \( x = 0 \) is a regular singular point of the equation \( xy'' + (2 - x)y' - y = 0 \). Find two independent solutions using the method of Frobenius (solve the indicial equation and compute the first three nonzero terms in the power series expansion assuming that the first nonzero coefficient is equal to 1).

**Question 11.**

a) State a version of Green’s theorem in the plane.

b) Find the area enclosed by the curve \( \Gamma \) in the plane defined by the implicit equation \( x^{2/3} + y^{2/3} = a^{2/3} \) where \( a > 0 \).

**Hint:** Use the symmetry of the equation in both \( x \) and \( y \). Also establish that \( x = a^3 \cos^3 \theta, y = a^3 \sin^3 \theta, 0 \leq \theta \leq \pi/2, \) is a parametrization of \( \Gamma \) in the first quadrant, and use this parametrization.

**Question 12.**

a) State the divergence theorem.

b) Use the divergence theorem or otherwise evaluate

\[ \int \int_{\Sigma} \mathbf{F} \cdot d\mathbf{s}, \]

where

i) \( \mathbf{F} = (\sin z + x^3 + y, e^2 + y^3 + 2y, z^3 + z^2); \) and

ii) \( \Sigma \) consists of the part of the sphere \( x^2 + y^2 + z^2 = 4z \) outside the paraboloid \( P \) given by \( z = x^2 + y^2 \) and the part of \( P \) inside \( S \); this surface is oriented with the normal pointing away from the region enclosed between \( P \) and \( S \).