McGILL UNIVERSITY
DEPARTMENT OF MATHEMATICS & STATISTICS
PART A – PAPER ALPHA

INSTRUCTIONS

Date: Monday, August 17, 1998
Time: 9:00 A.M. - 1:00 P.M.
Room: BURN 1120

Students in Pure and Applied Mathematics do the following questions: Analysis, Linear Algebra, Complex Variables, Calculus and Differential Equations (8 questions).
Students in Statistics do the following questions: Analysis, Linear Algebra and Probability (8 questions).
Students writing the Extended Alpha do the following questions: Analysis, Linear Algebra and Extended Alpha Questions (8 questions).

ANALYSIS

1. State the mean value theorem of the differential calculus. Let $f$ be a function continuous on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$. If $f(0) = 0$ and $f'$ is decreasing on $(0, 1)$ prove that

(a) $\frac{f(x)}{x} \geq f'(x)$, for $0 < x < 1$;

(b) the function $g = (0, 1) \rightarrow \mathbb{R}$ defined by $g(x) = \frac{f(x)}{x}$ is decreasing on $(0, 1)$.

2. (a) If $0 \leq a_{n-1} \leq a_n$ for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} a_n$ converges prove that $\lim_{n \to \infty} na_n = 0$.

(b) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and $\sum_{n=1}^{\infty} b_n$ an absolutely convergent series. Prove that $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent.

(c) If $a_n > 0$ for all $n \in \mathbb{N}$, prove that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}$ converge or diverge together.

LINEAR ALGEBRA

1. Consider the space $V$ of all $2 \times 2$ real matrices with the usual addition and scalar multiplication. Consider the linear transformation $T : V \rightarrow V$ defined, for fixed matrices

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

by $T(X) = AXA + XB$. Describe the kernel and the range space of $T$. (Find bases of these subspaces.)
2. Consider the real inner product space $P_1(t)$ of algebraic polynomials of degree at most 1, with the inner product

$$(x, y) = \int_{-1}^{1} x(t)y(t)dt.$$ 

In this space consider the linear transformation $T$ defined by

$$T(x) = \frac{dx}{dt} + 2x.$$ 

(a) Using the Cayley-Hamilton theorem express $T^{-1}$ in terms of $T$.

(b) Find $T^*(2 + t)$, where $T^*$ denotes the adjoint of $T$.

**COMPLEX VARIABLES**

1. Evaluate

$$\int_{\gamma} \frac{dz}{(z - 4)(z^3 - 1)}$$

where $\gamma$ is the circle $|z| = 2$ traversed counterclockwise. Simplify your answer to the form $a + bi$, $a$, $b$ real.

2. Find the number of zeros of the polynomial $z^3 + z^2 + 3z + 16$ in the right half plane \{ $z : \text{Re} z > 0$ \}. Hint: Consider the boundary of a large half-disc $|z| \leq R$, Re $z \geq 0$.

**CALCULUS**

1. Compute $\int \int \exp \left( \frac{y - x}{y + x} \right) dxdy$ over the triangle with vertices $(0, 0), (1, 0), (0, 1)$.

**DIFFERENTIAL EQUATIONS**

1. Let $f$ and $g$ be real valued functions defined on an interval $[a, b]$. Suppose that $f$ is continuous, $g$ is differentiable, and

$$g'(x) \leq f(x)g(x) \text{ for all } x \in [a, b].$$

Prove that

$$g(x) \leq g(a) \exp \left( \int_{a}^{x} f(t)dt \right)$$

for all $x \in [a, b]$. 
PROBABILITY

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

   (a) Show that $\mu$ is $\sigma$-finite if and only if for each $A \in \mathcal{F}$, there exists $f \in L^1_+(\mu)$ such that $\{f > 0\} = A$.

   (b) Suppose that $\mu$ is $\sigma$-finite. A set $A \in \mathcal{F}$ is an atom of $\mu$ if $\mu(A) > 0$ and if $\mu(B) = 0$ or $\mu(A)$ whenever $B \in \mathcal{F}$ and $B \subset A$. Show that there can be only countable many atoms of $\mu$, provided we identify atoms $A$ and $B$ for which $\mu(A \Delta B) = 0$.

2. Let $X_1, X_2, X_3, \ldots$ be i.i.d. exponentially distributed random variables, with mean $1/\lambda$. Let

   $S_n = \begin{cases} X_1 + \cdots + X_n & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}$

   and define $N_t = \max\{n \geq 0 | S_n \leq t\}$, $t \geq 0$.

   (a) Show that for each $t > 0$, $N_t$ has a Poisson distribution with mean $\lambda t$.

   (b) Let $U_t = S_{N_t+1} - t$ and $V_t = t - S_{N_t}$ for $t \geq 0$. Show that $U_t$ and $V_t$ are independent for each $t \geq 0$, that $U_t \sim X_1$, and that $V_t \sim \min\{X_1, t\}$. ($\sim$ means "equivalent in distribution"). You may assume $\{N_t, t \geq 0\}$ has been shown to have stationary independent increments.

3. Let $\{X_n, n \geq 1\}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$, let $X$ be a r.v. on $(\Omega, \mathcal{F}, P)$, and let $p \geq 1$.

   (a) Define what is meant by saying $\{X_n, n \geq 1\}$ is uniformly integrable.

   (b) Suppose that $\{|X_n|^p, n \geq 1\}$ is uniformly integrable, and that $X_n \to X$ in probability. Show that $X \in L^p(P)$ and that $X_n \to X$ in $L^p(P)$.

   (c) Suppose that $\{X_n, n \geq 1\} \subset L^1(P)$ are such that $X_n \geq 0$ a.s. for all $n$, that $X \in L^1(P)$, that $X_n \to X$ in probability, and that $EX_n \to EX$. Show that $X_n \to X$ in $L^1(P)$.

4. For each case, give an example of a sequence $\{X_n, n \geq 0\}$ of random variables and a random variable $X$ (defined on the same probability space) such that

   (a) $X_n \to 0$ in $L^p$, but $X_n \not\to 0$ a.s.

   (b) $X_n \to 0$ a.s., but $X_n \not\to 0$ in $L^p$.

   (c) $X_n \to X$ in distribution but $X_n \not\to X$ in probability,

   where $p \geq 1$. In each case, justify your statement.
EXTENDED ALPHA QUESTIONS

1. Let \( f \) be a bounded function on a closed interval \([a, b]\). Define carefully but concisely what is meant by the statement "\( f \) is Riemann integrable on \([a, b]\)."

A function \( g \) on \([a, b]\) is called a "step function if there exists a partition \( a = s_0 < s_1 < s_2 \cdots < s_n = b \), and \( \alpha_j \in \mathbb{R} \), s.t. \( g(x) = \alpha_j \) if \( x \in [s_{j-1}, s_j] \), \( j = 1, \cdots, n \), \( g(a) = \alpha_1 \). Show that a step function is Riemann integrable.

Show that if \( f \) is Riemann integrable on \([a, b]\), then given \( \varepsilon > 0 \), there exists a step function \( g \) on \([a, b]\) such that

\[
\int_a^b |f(x) - g(x)| \, dx < \varepsilon.
\]

2. (a) Prove that the series \( \sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} \) converges uniformly on every bounded interval, but does not converge absolutely for any \( x \in \mathbb{R} \).

(b) If \( a_n = 1 \) when \( n = m^2 \) for some \( m \in \mathbb{N} \), and \( a_n = 0 \) otherwise find the radius of convergence of \( \sum_{n=1}^{\infty} a_n x^n \).

3. Let \( A \) be an arbitrary complex \( n \times n \) matrix. Suppose that there is a positive definite Hermitian matrix \( K \) such that \( A^* K = K A \). Prove that \( A \) has only real eigenvalues and \( n \) linearly independent eigenvectors. (Here \( A^* \) denotes the complex conjugate of the transpose of the matrix \( A \).)

4. (a) State a definition of the cross product \( A \times B \) of two vectors \( A, B \in \mathbb{R}^3 \).

(b) If \( \{e_1, e_2, e_3\} \) is any positively oriented orthonormal basis of \( \mathbb{R}^3 \) and \( A_i = A \cdot e_i, \ B_i = B \cdot e_i \) for \( i = 1, 2, 3 \), show that

\[
A \times B = \begin{vmatrix} e_1 & e_2 & e_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}.
\]

(c) If \( \{v_1, v_2, v_3\} \) is any basis of \( \mathbb{R}^3 \) and \( \alpha_i = A \cdot v_i, \ \beta_i = B \cdot v_i, \ i = 1, 2, 3 \), show that

\[
A \times B = \frac{1}{V} \begin{vmatrix} v_1 & v_2 & v_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}.
\]

where \( V = \det(v_1, v_2, v_3) \) is the determinant of the \( 3 \times 3 \) matrix whose rows are \( v_1, v_2, v_3 \).