Date: Monday, August 18, 1997
Time: 9:00 A.M. - 1:00 P.M.

Instructions

1. All questions are compulsory
2. Satisfactory performance on questions 1 - 4 on Paper $\alpha$ is necessary in addition to an overall pass.

1. Let $T$ be a linear operator on a vector space $V$ of finite dimension. Let $W$ be a subspace of $V$ invariant under $T$, i.e. $T(W) \subset W$. Let $T_W : W \to W$ be the restriction of $T$ to $W$.

   (i) Show that the characteristic polynomial of $T_W$ divides the characteristic polynomial of $T$.

   (ii) Show that the minimum polynomial of $T_W$ divides the minimum polynomial of $T$.

2. Suppose the characteristic polynomial of $T : V \to V$ is $\Delta(t) = f_1(t)^{n_1} f_2(t)^{n_2} \ldots f_r(t)^{n_r}$ where the $f_i(t)$ are distinct monic irreducible polynomials. Let $V = W_1 \oplus \ldots \oplus W_r$ be the primary decomposition of $V$ into $T$-invariant subspaces. Show that $f_i(t)^{n_i}$ is the characteristic polynomial of the restriction of $T$ to $W_i$.

3. Prove, or disprove by counter example, each of the following statements about a sequence $(a_k)$ of real numbers:

   (i) If $\lim_{n \to \infty} n \left( \frac{a_{n+1}}{a_n} - 1 \right) = \ell < 1$, then $\sum_{1}^{\infty} a_n$ diverges;

   (ii) If $\sum_{1}^{\infty} a_n$ is a convergent series of positive numbers then $\lim_{n \to \infty} n a_n = 0$;

   (iii) If $\sum_{1}^{\infty} a_n$ is a divergent series of positive numbers, then $\sum_{1}^{\infty} \frac{a_n}{1 + a_n}$ is divergent.
4. Let \( f \) be a continuously differentiable function on \([0, 1]\).
   
   (i) If \( M_k, m_k \) are the least upper and greatest lower bounds of \( f' \) in \( \left[ \frac{k}{n}, \frac{k+1}{n} \right] \), 
   
   \( k = 0, 1, 2, \ldots, n - 1 \), show that for \( x \in \left[ \frac{k}{n}, \frac{k+1}{n} \right] \) 
   
   \[ m_k \left( x - \frac{k}{n} \right) \leq f(x) - f \left( \frac{k}{n} \right) \leq M_k \left( x - \frac{k}{n} \right). \]

   (ii) Obtain the limit as \( n \to \infty \) of the expression 
   
   \[ n \left\{ \int_{0}^{1} f(x) - \frac{1}{n} \sum_{k=0}^{n-1} f \left( \frac{k}{n} \right) \right\}. \]

5. Using the divergence theorem or otherwise, find 
   
   \[ \iint_{\partial D} (x^2 + x, y^2, z^2 + z) \cdot \vec{n}ds \]

   where \( \partial D \) is the boundary of the region \( D \) in \( \mathbb{R}^3 \),

   \[ D = \{(x, y, z) : 0 \leq z \leq 3/2, z^2/3 \leq x^2 + y^2 \leq 2z - z^2\}, \]

   \( ds \) is surface area, and \( \vec{n} \) is the outward normal on \( \partial D \).

6. Find the general solution of 
   
   \[ y'' + 6y' + 9y = \frac{1}{2} e^{-3x} + \sin x. \]

7. (a) For \( f(z) = \frac{1}{z^2 + z - 2} \), how many distinct series expansions are possible in integer powers of \( (z - 2) \), i.e. Taylor's or Laurent series?

   (b) Obtain in each case the general term, and state the region of convergence in each case.

8. Evaluate 
   
   \[ \oint_{C} \frac{-1 + \cos z}{z^2 \sin z} dz \]

   where \( C \) is the circle \(|z - 3| = 4\) in the complex plane.