1. All questions in Paper α are compulsory.

2. Satisfactory performance on questions 1-4 on Paper α is necessary in addition to an overall pass.

1. (a) Suppose that $S$ and $T$ are commuting linear operators on a finite dimensional complex vector space $V$ and that $S$ is diagonalizable. Show that there is a basis that diagonalizes $S$ and reduces $T$ to Jordan canonical form.

   (b) Show that if $S$ and $T$ are linear operators on a finite dimensional complex vector space, then $ST$ and $TS$ have the same characteristic polynomial.

2. (a) Let $T$ be a linear operator on the finite dimensional vector space $V$ and let $f(t)$ be a polynomial such that $f(T) = 0$. Suppose that $f(t) = g(t)h(t)$ with $g(t)$ and $h(t)$ having no common divisor. Show that $V = \text{Ker}(g(T)) \oplus \text{Ker}(h(T))$.

   (b) Let $T$ be a linear operator on an $n$ dimensional vector space. Show that $\text{Ker}(T^n) \cap \text{Im}(T^n) = \{0\}$.

3. Let $f$ be a bounded real valued function on $[a, b]$.

   (a) Define the upper (and lower) Darboux sums $U(f; P)$ (resp. $L(f; P)$) of $f$ corresponding to a partition $P$ of $[a, b]$.

   (b) Suppose $f^+(x) = \max(f(x), 0)$. Show that $U(f^+; P) - L(f^+; P) \leq U(f; P) - L(f; P)$.

   (c) Show that if $f$ is Riemann integrable so is $f^+$. 
(d) If \( f \) and \( g \) are Riemann integrable on \([a, b]\), prove that \( F \) and \( G \) are Riemann integrable where \( \forall x, F(x) = \max(f(x), g(x)) \) and \( G(x) = \min(f(x), g(x)) \).

4. Suppose \( \{f_n\} \) is a pointwise decreasing sequence of real valued continuous functions on a closed bounded interval \([a, b]\) such that \( \forall x, \lim_{n \to \infty} f_n(x) = 0 \). Prove that the sequence \( \{f_n\} \) tends to 0 uniformly on \([a, b]\). Give an example of an open interval \( I \) and a decreasing sequence \( \{g_n\} \) of continuous functions on \( I \) so that \( g_n(x) \) decreases to 0 for each \( x \in I \), BUT NOT UNIFORMLY.

5. Obtain the general solution of the differential equation
\[
x^2y'' - xy' + y = 2x
\]

6. Obtain the general solution of the differential equation
\[
3xy'' + y' - y = 0
\]

7. A non-zero scalar field \( \psi \) is such that \( \| \nabla \psi \|^2 = 3\psi \) and \( \nabla \cdot (\psi \nabla \psi) = 10\psi \). Evaluate
\[
\oiint_S \nabla \psi \cdot \hat{n} dS
\]
where \( S \) is the surface of the region in the first octant bounded by
\[
z = \sqrt{x^2 + y^2}, \quad z = \sqrt{1 - x^2 - y^2}, \quad y = x \quad \text{and} \quad y = \sqrt{3}x.
\]
Do this problem using both cylindrical and spherical coordinates.
Hint: \( \nabla \cdot (\psi \overline{F}) = \nabla \psi \cdot \overline{F} + \psi \nabla \cdot \overline{F} \).

8. Evaluate
\[
\oint_C \frac{-1 + \cos z}{z^2 \sin z} \, dz,
\]
where \( C \) is the circle |z - 3| = 4.