MCgill University

PART A EXAMINATION IN STATISTICS

THEORY PAPER

DEPARTMENT OF MATHEMATICS & STATISTICS

Date: 5 May 2003

Time: 13:00-17:00

INSTRUCTIONS

• Answer two questions out of Section P.

• Answer four questions out of Section S.

• All questions are weighted equally.

• Each question will be assessed independently by at least two members of the statistics group, and the final result determined after discussion within the Part A Exam Subcommittee.

• Your best 2 answers from Section P and best 4 answers from Section S will be used for the purpose of grading.

• Your answers to other questions may be used as assessment aids.

• Good luck!

This exam comprises the cover and 9 questions on 4 pages.
Section P: Answer two questions out of questions P1 to P3.

P1. (a) State the Monotone Convergence Theorem.
(b) Give a precise statement of the Strong Law of Large Numbers (SLLN).
(c) Let $X_1, X_2, \cdots$, be a sequence of independent and identically distributed random variables with $E(X_1^+) = \infty$, and $E(X_1^-) < \infty$. Define $S_n = \sum_{i=1}^{n} X_i$. Show that

$$\frac{S_n}{n} \to \infty \text{ a.s. as } n \to \infty.$$ 

[Hint: For any random variable $Y$, and constant $k$, we can write $Y_k = Y I_{[Y \leq k]}(\omega) + Y I_{[Y > k]}(\omega)$, where $I_A(\omega) = 1$ if $\omega \in A$, and $= 0$ if $\omega \in A^c$.]

P2. (a) State the Borel-Cantelli lemma for an arbitrary sequence of events $\{E_n\}$.
(b) Let $\{X_n\}$ be an arbitrary sequence of random variables. Show that it is always possible to find a sequence of constants, $\{a_n\}$, such that $\frac{X_n}{a_n} \to 0$ a.s.

[Hint: Consider $\sum_{n=1}^{\infty} P(|\frac{X_n}{a_n}| > \frac{1}{n})$, for suitable $a_n$.]
(c) State the Borel 0-1 law for a sequence of independent events.

P3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : (\Omega, \mathcal{F}, P) \to (\mathcal{R}, \mathcal{B})$ be a random variable. Suppose $\mu$ is the probability measure induced on $(\mathcal{R}, \mathcal{B})$ by $X$. Show that for any measurable function $f : (\mathcal{R}, \mathcal{B}) \to (\mathcal{R}, \mathcal{B})$,

$$\int_{\Omega} f(X(\omega)) P(d\omega) = \int_{\mathcal{R}} f(x) \mu(dx)$$

[Hint: Start with indicator functions, i.e. $f(x) = I_A(x)$.]
Section S: Answer four questions out of questions S1 to S6.

S1. (a) Let $X = [X_1, \ldots, X_n]'$ be a random sample from the family of (possibly multivariate) densities $\{f_{\theta} : \theta \in \Theta\}$. Suppose that $T = T(X) = [T_1(X), \ldots, T_q(X)]'$ is $\theta$-minimal sufficient. Suppose further that some non-constant function of $T$ with finite mean is $\theta$-ancillary. Use Basu's theorem to show that $T$ is not $\theta$-complete.

(b) Let $X|N \sim \text{Binomial}(N, p)$, where $N$ is a random variable such that $\text{Prob}[N = n] = p_n$, $n = 1, 2, \ldots$ and $\mathbb{E}[N] < \infty$. The $p_n$ values are functionally independent of $p$.

i. Show that $(X, N)$ is $p$-minimal sufficient.

ii. Show that $(X, N)$ is not $p$-complete.

(c) Let $X = [X_1, \ldots, X_n]$ be a random sample from a $U(\theta, \theta + 1)$ distribution, the uniform distribution on $(\theta, \theta + 1)$. Let $[X_{(1)}, \ldots, X_{(n)}]$ be the order statistics of $X$.

i. Show that $(X_{(1)}, X_{(n)})$ is $\theta$-minimal sufficient.

ii. Show that $(X_{(1)}, X_{(n)})$ is not $\theta$-complete.

(d) In the setting of subquestion (a) but without assuming that $T$ is $\theta$-sufficient (and without using Basu's theorem), prove that $T$ is not $\theta$-complete.

S2. (a) Show that for random variables $X$ and $Y$ with $\mathbb{E}[|Y|] < \infty$,

i. $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$, and

ii. $\text{Var}[Y] = \mathbb{E}[\text{Var}[Y|X]] + \text{Var}[\mathbb{E}[Y|X]]$.

(b) Let $X = [X_1, X_2, \ldots, X_n]'$ be a random sample from density $f_{\theta}(x)$. Show that under squared error loss, the Bayes' estimator of $\theta$ is $\mathbb{E}[\theta|X]$. (Note: Under loss function optimality, a Bayes' estimator minimizes posterior expected loss.)

(c) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed Bernoulli($p$) random variables. Suppose that $p$ has a Beta($\alpha, \beta$) prior, $\alpha, \beta > 1$.

i. Show that the posterior distribution of $\theta$ is again Beta, and interpret the information contributed by the prior distribution.

ii. Describe how you would determine a shortest length 95% credibility interval for $p$.  

Continued
S3. Let $X$ have the Pareto distribution with density

$$f_X(x) = \frac{\beta \alpha^\beta}{x^{\beta+1}} 1[x > \alpha].$$

for fixed $\alpha > 0$ and for $\beta > 0$.

(a) Express $f_X$ as a a member of a General Exponential Family with an appropriate canonical parameterization.

(b) Show that for $\epsilon > 0$,

$$\text{Prob}(|\log X - \log \alpha - \beta^{-1}| \geq \epsilon) \leq \frac{1}{\beta^2 \epsilon^2}.$$

Cite without proof any result you might need to establish this inequality.

(c) Let $W$ and $Z$ be random variables such that $W \leq Z$ with probability 1. Show that $\text{Prob}(W \geq \epsilon) \leq \text{Prob}(Z \geq \epsilon)$.

(d) Show that as $\beta \to \infty$,

$$\log X \overset{p}{\to} \log \alpha,$$

where $\overset{p}{\to}$ indicates convergence in probability.

*Hint:* Use $|a| \leq |a - b| + |b|$, (b) and (c).

S4. Let $X = [X_1, \ldots, X_n]'$ be a random sample from a $N(\mu, \sigma^2)$ distribution, $\mu \in \mathbb{R}$, $\sigma^2$ known. We wish to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Show that there is no uniformly most powerful (UMP) test of these hypotheses. You may use without proof the form of the critical region of the UMP test for one-tailed hypotheses.

S5. Let $X = [X_1, \ldots, X_n]'$ be a random sample from a $U(0, \theta)$ distribution, $\theta > 0$, and let $Y = \max_{1 \leq i \leq n} X_i$. Let $C_1(X) = [Y + a, Y + b]$, $0 \leq a < b$, and $C_2(X) = [cY, dY]$, $1 \leq c < d$, where $a, b, c, d$ are specified constants. Find the confidence coefficients of $C_1(X)$ and $C_2(X)$ respectively. (*Hint:* Consider the distribution of $Y/\theta$.)

*Continued*
S6. The Cramér-Rao Theorem can be stated as follows: Let $X_1, \ldots, X_n$ be a sample with pdf $f_\theta(x)$, $\theta \in \Theta \subset \mathbb{R}$, and let $W(X) = W(X_1, \ldots, X_n)$ be any estimator that satisfies condition A. Suppose that the joint pdf $f_\theta(x)$ satisfies condition B. Then

$$\text{Var}_\theta [W(X)] \geq \frac{\left( \frac{\partial}{\partial \theta} \mathbb{E}_\theta [W(X)] \right)^2}{\mathbb{E}_\theta \left[ \left( \frac{\partial}{\partial \theta} \log f_\theta(X) \right)^2 \right]}.$$ 

(a) Write down condition A and condition B.

(b) One proof of the Cramér-Rao Theorem relies on the Cauchy-Schwartz Inequality $(\text{Cov}[Y, Z])^2 \leq (\text{Var}[Y]) (\text{Var}[Z])$. Indicate what random variables should be chosen for $Y$ and $Z$ in order to carry out the proof.

(c) Under conditions A and B, use the condition for equality in the Cauchy-Schwartz Inequality to show that a necessary and sufficient condition for $W(X)$ to be UMVU is that

$$U(\theta|X) = a(\theta) W(X) + b(\theta) \quad \text{w.p. 1}$$

where $a$ and $b$ are functions of $\theta$ and $U(\theta|X)$ is the score at $\theta$ given by

$$U(\theta|X) = \frac{\partial}{\partial \theta} \log f_\theta(X).$$

(d) For both of the following distributions, let $X_1, \ldots, X_n$ be a random sample. Is there a function of $\theta$, say $g(\theta)$, for which there exists a UMVUE? If so, find it; if not, show why not. (You may assume that conditions A and B are satisfied in both cases.)

(i) $f_\theta(x) = \theta x^{\theta-1} 1[0 < x < 1]$, for $\theta > 0$.

(ii) $f_\theta(x) = \frac{\log \theta}{\theta - 1} x^\theta 1[0 < x < 1]$, for $\theta > 0$.

End of Theory Paper.