McGill University
Department of Mathematics and Statistics
Part A Ph.D. Preliminary Examination
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Do three of questions [1] to [4],
three of questions [5] to [8],
and three of [9] to [12].
Do three of questions [1] to [4]:

[1] We are given vector spaces $U$, $V$ and $W$ of respective dimensions 3, 2 and 4, and linear mappings $U \xrightarrow{S} V \xrightarrow{T} W$. We have the bases

$$e = (e_1, e_2, e_3)$$ and $$f = (f_1, f_2, f_3)$$ of $U$, 

$$g = (g_1, g_2)$$ and $$h = (h_1, h_2)$$ of $V$, 

and

$$r = (r_1, r_2, r_3, r_4)$$ and $$s = (s_1, s_2, s_3, s_4)$$ of $W$.

Furthermore, we are given the following facts:

$$
\begin{align*}
    f_1 &= e_1 - e_2 + e_3 & h_1 &= g_1 + 3g_2 & s_1 &= -r_1 - r_2 + 2r_3 + 3r_4 \\
    f_2 &= e_2 + 2e_3 & h_2 &= 5g_1 - 6g_2 & s_2 &= r_1 + 7r_2 - 2r_4 \\
    f_3 &= e_1 & h_3 &= 5g_1 - 6g_2 & s_3 &= r_2 - r_3 + 9r_4 \\
    & & s_4 &= r_1 + r_2 - r_3 + 2r_4
\end{align*}
$$

$$
\begin{align*}
    S(e_1) &= g_1 - 2g_2 & T(h_1) &= s_1 + 2s_2 - s_3 + 7s_4 \\
    S(e_2) &= -2g_1 + g_2 & T(h_2) &= -s_1 + s_2 - 8s_3 + 6s_4 \\
    S(e_3) &= -g_1 + 4g_2
\end{align*}
$$

Without any computation, give the matrix of the composite $T \circ S : U \longrightarrow W$ relative to the bases $f$ of $U$ and $r$ of $W$ in the form of a product of matrices; each factor should either be given with numerical entries, or it should be the inverse of a matrix with numerical entries.

**Remark** The solution should involve no computation at all; no computation should be used for writing down the matrices, and no inverses or matrix products should be computed. The answer should demonstrate a general method that may be used in the case of a similar problem involving large dimensions when all the computation is left to be done by a computer.
[2] Let \( V \) be a finite dimensional inner product space over the complex scalars, \( T \) a linear operator on \( V \).

(i) Define the adjoint of \( T \), and define when \( T \) is said to be normal; the definition of "normal" should be given in terms of the concept of the adjoint operator.

(ii) Let \( B \) be an orthonormal basis of \( V \), and assume that the matrix of \( T \) relative to \( B \) is diagonal. Prove that \( T \) is normal. Make sure your argument is complete.

[3] Let \( T \) be a linear operator on a finite dimensional vector space \( V \). Prove that \( \ker(T^2) = \ker(T) \) if and only if \( \im(T^2) = \im(T) \).

[4] The linear operator \( T \) on \( \mathbb{R}^4 \) is given by its matrix \( A \) relative to the standard basis; \( A \) is the matrix

\[
A = \begin{pmatrix}
1 & 1 & -3 & -2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Determine a basis \( B \) of \( \mathbb{R}^4 \) for which the matrix \( B=[T]_B \) of \( T \) relative to \( B \) is in Jordan form, and determine the matrix \( B \) as well.
Do three of questions [5] to [8]:

[5] Suppose that \( f \) and \( f_n \ (n=1, 2, \ldots) \) are continuous functions defined on \([0, 1]\) such that

\[
f_n(x) \leq f_{n+1}(x) \quad \text{for all } n \text{ and } x \in [0, 1],
\]

and

\[
\lim_{n \to \infty} f_n(x) = f(x) \quad \text{for each } x \in [0, 1].
\]

Prove that the \( f_n \) converge uniformly to \( f \).

[6] (i) Using, among others, the formulas for the derivatives of the sine and cosine functions, deduce that \( \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} \).

(ii) State precisely the theorem on the term-by-term Riemann integration of a function series.

(iii) Using (i) and (ii), develop a power series for \( \arctan(x) \) around \( x=0 \), and determine its radius of convergence. Justify all your steps.

(iv) What happens at the endpoints of the interval of convergence of the power series of (iii)? Where does the power series represent the function \( \arctan(x) \)?
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(iv) What happens at the endpoints of the interval of convergence of the power series of (iii)? Where does the power series represent the function \( \arctan(x) \)?
Do three of questions [9] to [12]:

[9]  
(i)  State the Divergence Theorem.  
(ii) Use the Divergence Theorem to evaluate the surface integral

\[ \int \int_S (x^2 + y^2) \, ds \]

where \( S \) is the sphere \( x^2 + y^2 + z^2 = 1 \).

[10]  Solve the following initial value problem:

\[ \frac{dx_1}{dt} = x_1 + 2x_2 + 3x_3 \]
\[ \frac{dx_2}{dt} = x_2 - 2x_3 \]
\[ \frac{dx_3}{dt} = x_3 \]

\[ x_1(0) = 1, \ x_2(0) = 2, \ x_3(0) = 3 \]

(ii) Prove that all zeros of $\cos(z)$ are real.

(iii) Calculate $\int \tan(z)$.

\[ |z| = 2 \]

[12] (i) State the Maximum-Modulus Theorem.

(ii) Let $\alpha$ be a complex number with $|\alpha| < 1$, and let

\[ B_\alpha(z) \overset{\text{def}}{=} \frac{z - \alpha}{1 - \bar{\alpha} z} \].

Verify that $B_\alpha$ is analytic in the region $|z| \leq 1$, vanishes at $\alpha$, and

\[ |B_\alpha(z)| = 1 \text{ for } |z| = 1 \].

(iii) Suppose that $f(z)$ is analytic in the region $|z| \leq 1$, vanishes at some $\alpha$ ($|\alpha| < 1$), and $|f(z)| \leq 1$ for $|z| = 1$. Show that $|f(z)| \leq |B_\alpha(z)|$ for $|z| \leq 1$. 

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