MATH 557 - MID-TERM EXAMINATION 2008

Marks can be obtained by answering all questions.

1. Consider the probability model

$$f_{X|\theta,\sigma}(x|\theta,\sigma) = \exp\left\{-\left(\frac{x-\theta}{\sigma}\right)^4 - \kappa(\theta,\sigma)\right\} \qquad -\infty < x < \infty$$

for $\theta \in \mathbb{R}$ and $\sigma > 0$, for some function $\kappa(.,.)$.

(a) Is this probability model an Exponential Family distribution ? Justify your answer.

6 Marks

(b) Find a (possibly multivariate) sufficient statistic for $(\theta,\sigma)^{\mathsf{T}}$ based on a random sample of size n, $X_1,\ldots,X_n$.

6 Marks

2. Let $X_1,\ldots,X_n$ be a random sample of size n from a $Normal(\theta,\theta^2)$ distribution, for parameter $\theta > 0$.

(a) Find a minimal sufficient statistic for θ (and demonstrate minimal sufficiency for the statistic you find).

6 Marks

(b) Is the statistic from part (a) complete ? Justify your answer.

6 Marks

3. This question concerns estimation of parameter λ, the expected value of a $Poisson(\lambda)$ distribution, from a random sample $X_1,\ldots,X_n$ from that distribution.

(a) Derive the Bayes estimator, $\widehat{\lambda}_B(\underset{\sim}{X})$, of λ under a proper conjugate prior specification and squared-error loss

$$\mathcal{L}(\lambda(\underset{\sim}{x}),\lambda) = (\lambda(\underset{\sim}{x}) - \lambda)^2$$

and show that $\widehat{\lambda}_B(\underset{\sim}{X})$ can be written

$$\widehat{\lambda}_B(\underset{\sim}{X}) = w_n\overline{X}_n + (1-w_n)m$$

where $\overline{X}_n$ is the mean of $X_1,\ldots,X_n$, m is the mean of the prior distribution, and $0 \le w_n \le 1$ is a constant function of n.

10 Marks

(b) In a decision problem concerned with estimating parameter θ, the risk, $R_\delta(\theta)$, associated with decision $\delta(\underset{\sim}{X})$ for loss function $\mathcal{L}$ is the expected loss associated with $\delta(\underset{\sim}{X})$,

$$R_\delta(\theta) = \mathrm{E}_{f_{\underset{\sim}{X}|\theta}}\left[\mathcal{L}(\delta(\underset{\sim}{X}),\theta)\right] = \int_{\mathcal{X}} \mathcal{L}(\delta(\underset{\sim}{x}),\theta) f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta)\, d\underset{\sim}{x}.$$

Consider a decision problem relating to the estimation of θ. An estimator of θ, denoted $T(\underset{\sim}{X})$ say, is termed **inadmissible** if

$$R_T(\theta) \geq R_{T_0}(\theta) \qquad \text{for all } \theta \in \Theta$$

and $R_T(\theta) > R_{T_0}(\theta)$ for at least one $\theta \in \Theta$, where $T_0(\underset{\sim}{X})$ is some other estimator of θ.

In the $Poisson(\lambda)$ model from part (a), show that estimators of the form

$$T(\underset{\sim}{X}) = a\overline{X}_n + b$$

for $a > 1$ are inadmissible if $\mathcal{L}$ is squared-error loss.

6 Marks