MATH 557 - EXERCISES 1

These exercises are not for assessment

1 Let $\underset{\sim}{X} = (X_1, \ldots, X_n)^\mathsf{T}$ be a random vector with joint density in $\mathcal{F}_k = \{f_i(\underset{\sim}{x}) : i = 0, \ldots, k\}$, so that $\mathcal{F}_k$ is parameterized by index $i \in \{0, \ldots, k\}$. Assume that the densities in $\mathcal{F}_k$ have common support.

(a) Show that the statistic

$$T(\underset{\sim}{X}) = \left(\frac{f_1(\underset{\sim}{X})}{f_0(\underset{\sim}{X})}, \ldots, \frac{f_k(\underset{\sim}{X})}{f_0(\underset{\sim}{X})} \right)^\mathsf{T}$$

is minimal sufficient for i.

(b) Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ be a family of densities with common support, and suppose that

- $f_i \equiv f_{\theta_i} \in \mathcal{F}$, θ_i distinct, for $i = 0, \ldots, k$ and
- $T(\underset{\sim}{X})$ as defined in part (a) is sufficient for θ.

Show that $T(\underset{\sim}{X})$ is minimal sufficient for θ.

(c) Show that 1-1 functions of minimal sufficient statistics are minimal sufficient statistics.

(d) Use the results from parts (b) and (c) to show that the sample mean $\overline{X}$ is minimal sufficient for β if $\underset{\sim}{X}$ is a random sample from an Exponential distribution with expectation $\beta > 0$.

2 Suppose that $X_1, \ldots, X_n$ are a random sample from a $Multinomial(3, \underset{\sim}{\theta})$ distribution defined by the probabilities

$$\Pr[X_i = j] = \theta_j \qquad j = 1, 2, 3$$

and zero otherwise, where $0 < \theta_1, \theta_2, \theta_3 < 1$, and $\theta_1 + \theta_2 + \theta_3 = 1$.

(a) Find a (possibly vector-valued) sufficient statistic for $\underset{\sim}{\theta}$.

(b) Find the (joint) pmf of the sufficient statistic.

(c) FInd the form of the maximum likelihood estimator for $\underset{\sim}{\theta}$

3 Consider the location family pdf with standard member the $Cauchy$ distribution

$$f_{X|\theta}(x|\theta) = \frac{1}{\pi} \frac{1}{1 + (x - \theta)^2} \qquad -\infty < x < \infty$$

for location parameter $\theta \in \Theta \equiv \mathbb{R}$.

(a) Derive the *score equation* for θ defined for a random sample $X_1, \ldots, X_n$ from this pdf by

$$\frac{\partial l(\theta|\underset{\sim}{x})}{\partial \theta} = 0$$

where $l(\theta|\underset{\sim}{x}) = \log L(\theta|\underset{\sim}{x}) = \log f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta)$

(b) Using a computer package, plot the log-likelihood function $l(\theta|\underset{\sim}{x})$ for a suitable range of θ for the following observed x values:

$$7.36 \; 5.14 \; 3.71 \; 3.15 \; 6.00 \; 6.38 \; 1.34 \; 6.73$$

and hence find the maximum likelihood (ML) estimate.

4 Carry out a simulation study to examine the sampling distribution of the maximum likelihood estimator $\widehat{\theta}(\underset{\sim}{X})$ in the Cauchy location family example in the previous problem.

For example, in `R`:

- Produce $N = 5000$ simulated data sets of size $n = 8$, using a specific value of θ, and using the random number generation function `rcauchy`.
- For each simulated data set, use pointwise evaluation of the likelihood (or the function `optimize`) to evaluate the ML estimate in each case.
- Display using a histogram the distribution of the N stored ML estimates.

The sample median is an alternative estimator of θ. Repeat the computations above using this alternative estimator.

5 Suppose that $X_1 \sim Binomial(n_1, \theta_1)$ and $X_2 \sim Binomial(n_2, \theta_2)$ be independent random variables. Derive the maximum likelihood estimator of the odds ratio ψ defined by

$$\psi = \frac{\theta_1/(1-\theta_1)}{\theta_2/(1-\theta_2)}$$

Hint : write down the likelihood in terms of the 1-1 reparameterization

$$(\theta_1, \theta_2) \longrightarrow (\phi, \psi)$$

for some appropriately chosen parameter ϕ.