# MATH 557 - ASSIGNMENT 3
## SOLUTIONS

1 (a) To find the UMP test, consider

$$\begin{aligned} H_0 &: \theta = 1 \\ H_1 &: \theta = \theta_1 \end{aligned}$$

for $\theta_1 > 1$. By Neyman-Pearson, the rejection region is constructed by looking at

$$\frac{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta_1)}{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|1)} = \frac{\prod_{i=1}^{n} \theta_1 (1-x_i)^{\theta_1 - 1}}{1} = \theta_1^n \{T(\underset{\sim}{x})\}^{\theta_1 - 1}$$

where $T(\underset{\sim}{x}) = \prod_{i=1}^{n}(1-x_i)$. Hence the rejection region is defined by

$$\theta_1^n \{T(\underset{\sim}{x})\}^{\theta_1 - 1} > k \qquad \text{or equivalently} \qquad T(\underset{\sim}{x}) > k_1$$

where the requirement

$$\Pr[T(\underset{\sim}{X}) \in \mathcal{R}_T | \theta = 1] = \Pr[T(\underset{\sim}{X}) > k_1 | \theta = 1] = \alpha$$

determines $k_1$ for any $\alpha$. To simplify further

$$\prod_{i=1}^{n}(1-X_i) > k_1 \qquad \Longleftrightarrow \qquad -\sum_{i=1}^{n} \log(1-X_i) < -\log k_1 = c$$

say. Now, if $\theta = 1$, the data are uniformly distributed on (0,1). Also, if $X \sim \text{Uniform}(0,1)$, then $1 - X \sim \text{Uniform}(0,1)$, and

$$-\log(1-X) \sim \text{Exponential}(1)$$

Therefore the critical region is defined by

$$\Pr[T(\underset{\sim}{X}) > k_1 | \theta = 1] = \Pr[V < c | \theta = 1] = \alpha$$

where

$$V = -\log T(\underset{\sim}{X}) = -\sum_{i=1}^{n} \log(1-X_i) \sim \text{Gamma}(n,1).$$

Thus $c$ is the $\alpha$ quantile of the Gamma$(n,1)$ distribution. This is the UMP test for any $\theta_1 > 1$, so it is the UMP test for the required hypotheses.

4 MARKS

(b) Under $H_1$, the ML estimate of $\theta$ is

$$\widehat{\theta} = \underset{\theta \in \mathbb{R}^+}{\text{argmax}}\, \theta^n \{T(\underset{\sim}{x})\}^{\theta - 1} = -\frac{n}{\log T(\underset{\sim}{x})} = -\frac{n}{\sum_{i=1}^{n} \log(1-X_i)} = -\frac{n}{\log T(\underset{\sim}{x})}$$

Thus the LRT is based on the rejection region $\mathcal{R}_{\underset{\sim}{X}}$ defined by

$$\lambda_{\underset{\sim}{X}}(\underset{\sim}{x}) = \frac{L(1|\underset{\sim}{x})}{L(\widehat{\theta}|\underset{\sim}{x})} = \frac{1}{\widehat{\theta}^n \{T(\underset{\sim}{x})\}^{\widehat{\theta} - 1}} \leq k$$

which is equivalent to

$$n \log \widehat{\theta} + (\widehat{\theta} - 1) \log T(\underset{\sim}{x}) \geq -\log k$$

or

$$-n \log(-\log T(\underset{\sim}{x})) - \log T(\underset{\sim}{x}) \geq -\log k - n \log n + n$$

which may be written

$$-n \log V + V \geq c$$

where $V \sim \text{Gamma}(n, 1)$ as above. To solve this for $c$ requires numerical steps.

4 MARKS

2 Can use the Karlin-Rubin theorem in both cases.

(a) The likelihood ratio for $\theta_1 < \theta_2$ for this model is

$$\lambda(\underset{\sim}{x}) = \frac{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta_2)}{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta_1)} = \frac{\theta_1^n}{\theta_2^n} \exp\left\{T(\underset{\sim}{x})\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\right\}$$

which is an increasing function of $T(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i$. Thus the rejection region takes the form

$$\mathcal{R} \equiv \left\{\underset{\sim}{x} : T(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i > t_0\right\}$$

To find $t_0$, we need to solve

$$\Pr[T(\underset{\sim}{X}) > t_0 \mid \theta_0] = \alpha.$$

Here $T(\underset{\sim}{X}) \sim \text{Gamma}(n, 1/\theta)$, so $t_0$ is the $1 - \alpha$ quantile of this distribution.

4 MARKS

(b) The likelihood ratio for $\theta_1 < \theta_2$ for this model is

$$\lambda(\underset{\sim}{x}) = \frac{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta_2)}{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta_1)} = \frac{\theta_1^{n/2}}{\theta_2^{n/2}} \exp\left\{\frac{T(\underset{\sim}{x})}{2}\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\right\}$$

which is an increasing function of $T(\underset{\sim}{x}) = \sum_{i=1}^{n} (x_i - 1)^2$. Thus the rejection region takes the form

$$\mathcal{R} \equiv \left\{\underset{\sim}{x} : T(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i > t_0\right\}$$

To find $t_0$, we need to solve

$$\Pr[T(\underset{\sim}{X}) > t_0 \mid \theta_0] = \alpha.$$

Here under the assumption $\theta = \theta_0$,

$$\frac{T(\underset{\sim}{X})}{\theta_0} \sim \chi_n^2 \equiv \text{Gamma}(n/2, 1/2)$$

so

$$\Pr[T(\underset{\sim}{X}) > t_0 \mid \theta_0] = \Pr[T(\underset{\sim}{X})/\theta_0 > t_0/\theta_0 \mid \theta_0] = \alpha.$$

implies that $t_0 = \theta_0 q_{n,1-\alpha}$, where $q_{n,1-\alpha}$ is the $1 - \alpha$ quantile of the Chisquared distribution with $n$ degrees of freedom.

8 MARKS

3 Again using the Karlin-Rubin Theorem: The likelihood ratio for $\theta_1 < \theta_2$ for this model is

$$\lambda(\underset{\sim}{x}) = \frac{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta_2)}{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta_1)} = \left(\frac{\theta_2}{\theta_1}\right)^{T(\underset{\sim}{x})} \exp\left\{-n(\theta_2 - \theta_1)\right\}$$

where $T(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i$. In this case, under $\theta = 2$,

$$T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i \sim \text{Poisson}(2n)$$

Thus the distribution of $T(\underset{\sim}{X})$ is discrete. A randomized test takes the form

$$\phi^{\star}_{\mathcal{R}}(\underset{\sim}{x}) = \begin{cases} 1 & T(\underset{\sim}{x}) > c \\ \gamma & T(\underset{\sim}{x}) = c \\ 0 & T(\underset{\sim}{x}) \leq c \end{cases}$$

where $c$ is the largest integer such that

$$\Pr[T(\underset{\sim}{X}) > c] \leq 0.05$$

and $\gamma$ is selected so that

$$\Pr[T(\underset{\sim}{X}) > c] + \gamma \Pr[T(\underset{\sim}{X}) = c] = 0.05$$

In the example, $n = 6$, and $T(\underset{\sim}{x}) = 18$, and by calculation $c = 18$

$$\Pr[T(\underset{\sim}{X}) > 18] = 0.0374 \qquad \Pr[T(\underset{\sim}{X}) = 18] = 0.0255$$

so that

$$\gamma = \frac{0.05 - \Pr[T(\underset{\sim}{X}) > 18]}{\Pr[T(\underset{\sim}{X}) = 18]} = \frac{0.05 - 0.0374}{0.0255} = 0.494$$

In this case, the hypothesis is rejected with probability $\gamma = 0.494$ as $T(\underset{\sim}{x}) = 18$.

4 MARKS