MATH 557 - ASSIGNMENT 2
SOLUTIONS

1 (a) The joint pdf for $X_1, \ldots, X_n$ is

$$f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta) = \frac{1}{\theta^n} \exp\left\{-\frac{1}{\theta}\sum_{i=1}^n (x_i - \theta)\right\} I_{(x_{(1)},\infty)}(\theta) \qquad -\infty < \theta < \infty$$

where $x_{(1)} = \min\{x_1, \ldots, x_n\}$. Thus

$$\underset{\sim}{T}(\underset{\sim}{X}) = \left(\sum_{i=1}^n X_i, X_{(1)}\right)^{\mathsf{T}} = \left(T_1(\underset{\sim}{X}), T_2(\underset{\sim}{X})\right)^{\mathsf{T}},$$

say, is a sufficient statistic for θ. To demonstrate that this is a minimal sufficient statistic, note that for two vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$

$$\frac{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta)}{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{y}|\theta)} = \exp\left\{-\frac{1}{\theta}(T_1(\underset{\sim}{x}) - T_1(\underset{\sim}{y}))\right\} \frac{I_{(T_2(\underset{\sim}{x}),\infty)}(\theta)}{I_{(T_2(\underset{\sim}{y}),\infty)}(\theta)}$$

which does not depend on θ if and only if

$$T_1(\underset{\sim}{x}) = T_1(\underset{\sim}{y}) \qquad \text{and} \qquad T_2(\underset{\sim}{x}) = T_2(\underset{\sim}{y})$$

Hence $\underset{\sim}{T}(\underset{\sim}{X})$ is minimal sufficient. 4 MARKS

(b) Note first that for $i = 1, \ldots, n$,

$$X_i \stackrel{d}{=} Z_i + \theta$$

where $Z_i \sim Exp(1/\theta)$ constitute a random sample. Thus

$$T_1(\underset{\sim}{X}) = \sum_{i=1}^n X_i \stackrel{d}{=} \sum_{i=1}^n Z_i + n\theta$$

and

$$E_{f_{T_1(\underset{\sim}{X})}}[T_1(\underset{\sim}{X})] = \sum_{i=1}^n E_{f_{Z_i}}[Z_i] + n\theta = n\theta + n\theta = 2n\theta$$

by properties of the Exponential distribution. Secondly

$$T_2(\underset{\sim}{X}) = \min\{X_1, \ldots, X_n\} \stackrel{d}{=} \min\{Z_1, \ldots, Z_n\} + \theta$$

By results related to order statistics from MATH 556, we have that

$$F_{Z_{(1)}}(z) = 1 - \{1 - F_{Z_1}(z)\}^n = 1 - \exp\{-nz/\theta\} \qquad z > 0$$

so that $Z_{(1)} \sim Exp(n/\theta)$. Hence

$$E_{f_{T_2(\underset{\sim}{X})}}[T_2(\underset{\sim}{X})] = E_{f_{Z_{(1)}}}[Z_{(1)}] + \theta = \frac{\theta}{n} + \theta = \frac{(n+1)}{n}\theta.$$

Thus if we take function g to be

$$g(t_1, t_2) = \frac{1}{2n}t_1 - \frac{n}{n+1}t_2$$

and it follows that $\underset{\sim}{T}(\underset{\sim}{X})$ is **not** complete, as

$$E_{f_{\underset{\sim}{T}(\underset{\sim}{X})}}[g(T_1(\underset{\sim}{X}), T_2(\underset{\sim}{X}))] = 0$$

4 MARKS

2 (a) The expectation of random variable X from this distribution is

$$\mathrm{E}_{f_{X|\theta}}[X] = \int_{-1}^{1} x \frac{1+\theta x}{2}\, dx = \left[\frac{x^2}{4} + \theta \frac{x^3}{6} \right]_{-1}^{1} = \frac{\theta}{3}$$

By, for example, the strong law of large numbers

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} E_{f_{X|\theta}}[X] = \frac{\theta}{3}$$

and hence by the continuous mapping theorem

$$\widetilde{\theta}_n(\underset{\sim}{X}) = 3\overline{X}_n \xrightarrow{a.s.} \theta$$

as $n \longrightarrow \infty$, and is a consistent estimator of θ. 2 MARKS

(b) By the central limit theorem (which may be applied as the support of the pdf is bounded, and hence the variance of the distribution is finite), we have that

$$n^{1/2}(\overline{X}_n - \theta/3) \xrightarrow{d} Z \sim N(0, \sigma^2)$$

and thus by the Delta method applied with function $g(t) = 3t$, we have that

$$n^{1/2}(\widetilde{\theta}_n(\underset{\sim}{X}) - \theta) \xrightarrow{d} Z \sim N(0, 9\sigma^2)$$

Here

$$\mathrm{E}_{f_{X|\theta}}[X^2] = \int_{-1}^{1} x^2 \frac{1+\theta x}{2}\, dx = \left[\frac{x^3}{6} + \theta \frac{x^4}{8} \right]_{-1}^{1} = \frac{1}{3}$$

so that

$$\sigma^2 = \mathrm{Var}_{f_{X|\theta}}[X] = \frac{1}{3} - \frac{\theta^2}{9} = \frac{3-\theta^2}{9}$$

and the asymptotic variance of $\widetilde{\theta}_n(\underset{\sim}{X})$ is $(3-\theta^2)$. 2 MARKS

3 For any finite n, we have that

$$\sum_{i=1}^{n} I_{\{0\}}(X_i) \sim Binomial(n, \phi)$$

and thus, by results from lectures, properties of the Binomial distribution, and the Central Limit Theorem (CLT), the asymptotic variance of $\widetilde{\phi}_{n1}(\underset{\sim}{X})$ is

$$\sigma_1^2 = \phi(1-\phi)$$

Also by the CLT, we have for a random sample from a $Poisson(\theta)$ distribution that as $n \longrightarrow \infty$,

$$n^{1/2}(\overline{X}_n - \theta) \xrightarrow{d} Z \sim N(0, \theta).$$

Using the Delta method with mapping $g(t) = e^{-t}$, so that $\dot{g}(t) = -e^{-t}$, it follows that

$$n^{1/2}(\widetilde{\phi}_{n2}(\underset{\sim}{X}) - \phi) \xrightarrow{d} Z \sim N(0, e^{-2\theta}\theta),$$

so the asymptotic variance of $\widetilde{\phi}_{n2}(\underset{\sim}{X})$ is

$$\sigma_2^2 = -\phi^2 \log \phi$$

Thus the asymptotic relative efficiency is

$$\frac{\sigma_1^2}{\sigma_2^2} = \frac{(\phi - 1)}{\phi \log \phi}.$$

8 MARKS