MATH 557 - ASSIGNMENT 1
SOLUTIONS

1 We construct the joint pmf/pdf in each case, and inspect the required conditional pdf. Note that 1-1 transformations of the statistics are also sufficient.

(a) For the

$$f_{\underset{\sim}{X}|\alpha,\beta}(\underset{\sim}{x}|\alpha,\beta) = \left\{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right\}^n \left\{\prod_{i=1}^n x_i\right\}^{\alpha-1} \left\{\prod_{i=1}^n (1-x_i)\right\}^{\beta-1}$$

suggesting the sufficient statistic

$$\underset{\sim}{T}(\underset{\sim}{X}) = \left(\prod_{i=1}^n x_i, \prod_{i=1}^n (1-x_i)\right)^\top$$

and the result follows using the Neyman Factorization Theorem.

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(b) Writing $\lambda = \log\theta$, we realize that this is the $Poisson(\log\theta)$ model. Hence by properties of the Exponential Family

$$T(\underset{\sim}{X}) = \sum_{i=1}^n X_i$$

is a sufficient statistic for $\log\theta$.

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(c) We have

$$\begin{aligned} f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta) &= \frac{1}{\theta}^n & \theta < x_1, \ldots, x_n < 2\theta \\ &= \frac{1}{\theta}^n I_{(x_{(n)}/2, x_{(1)})}(\theta) \end{aligned}$$

suggesting the sufficient statistic

$$\underset{\sim}{T}(\underset{\sim}{X}) = \left(X_{(1)}, X_{(n)}\right)^\top$$

and the result follows using the Neyman Factorization Theorem.

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2 (a) We have

$$f_{\underset{\sim}{X}|\lambda}(\underset{\sim}{x}|\lambda) = \lambda^n \exp\left\{-\lambda\sum_{i=1}^n x_i\right\} = \lambda^n \exp\left\{-\lambda T(\underset{\sim}{x})\right\}$$

say, so that for two points $\underset{\sim}{x}$ and $\underset{\sim}{y}$ the ratio

$$\frac{f_{\underset{\sim}{X}|\lambda}(\underset{\sim}{x}|\lambda)}{f_{\underset{\sim}{X}|\lambda}(\underset{\sim}{y}|\lambda)} = \exp\left\{-\lambda\left(T(\underset{\sim}{x}) - T(\underset{\sim}{y})\right)\right\}$$

which is a constant if and only if $T(\underset{\sim}{x}) = T(\underset{\sim}{y})$. Therefore $T(\underset{\sim}{x})$ is minimal sufficient.

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(b) Given $x_{(1)}, \ldots, x_{(m)}$, we can construct the joint pdf for the order statistic data by noting that if $X_{(m)} = x_{(m)}$, then we have $X_{(r)} > x_{(m)}$ for the $n-m$ order statistics $X_{(r)}$, $r = m+1, \ldots, n$. Thus, as the "survivor" function takes the form $1 - F_{X|\lambda}(x|\lambda) = e^{-\lambda x}$, we have

$$f_{\underset{\sim}{X}|\lambda}(\underset{\sim}{x}|\lambda) = m!\binom{n}{m} \times \lambda^m \exp\left\{-\lambda \sum_{i=1}^{m} x_{(i)}\right\} \times \exp\left\{-(n-m)\lambda x_{(m)}\right\}$$

where the combinatorial term counts the number of possible arrangements of the random sample points. Thus a sufficient statistic is

$$T(\underset{\sim}{X}) = \sum_{i=1}^{m} X_{(i)} + (n-m)X_{(m)}$$

by the Neyman Factorization Theorem. 6 MARKS

3 We have for $t = 0, 1, \ldots,$

$$f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta) = \frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!}$$

and $T(\underset{\sim}{X}) \sim Poisson(n\theta)$ from distributional results, so that

$$f_{\underset{\sim}{X}|T(\underset{\sim}{X})}(\underset{\sim}{x}|t) = \frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta} / \prod_{i=1}^{n} x_i!}{(n\theta)^t e^{-n\theta}/t!} = \frac{t!}{x_1! \ldots, x_n!}\left(\frac{1}{n}\right)^t \qquad \underset{\sim}{x} \in A_t$$

and zero otherwise, where

$$A_t \equiv \{\underset{\sim}{x} : x_1 + \cdots + x_n = t\}.$$

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