MATH 557 - ASSIGNMENT 4

To be handed in not later than 5pm, 8th April 2008.
Please hand in during lectures, to Burnside 1235, or to the Mathematics Office Burnside 1005

1 Suppose that $X_1, \ldots, X_n \sim \text{Bernoulli}(\theta)$ are a random sample, where $0 < \theta < 1$, and suppose that $\tau(\theta) = \theta(1-\theta)$.

(a) Find the maximum likelihood estimator of $\tau(\theta)$, $\widehat{\tau}_n(\underset{\sim}{X})$.

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(b) Find large sample approximation to the distribution of $\widehat{\tau}_n(\underset{\sim}{X})$ for each $\theta \in (0,1)$.

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2 Suppose that $X_1, \ldots, X_n$ is a random sample from a distribution with pdf f_X, with

$$\mathrm{E}_{f_X}[X_i] = \mu \qquad \mathrm{Var}_{f_X}[X_i] = 1 \qquad \mathrm{Var}_{f_X}[X_i^2] = \gamma \qquad \mathrm{E}_{f_X}[X_i^4] < \infty$$

for $i = 1, \ldots, n$. Denote by

$$T_{1n}(\underset{\sim}{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 1 \qquad T_{2n}(\underset{\sim}{X}) = \overline{X}^2 - \frac{1}{n}$$

two estimators of $\tau(\mu) = \mu^2$. The *Asymptotic Relative Efficiency* (ARE) of T_{1n} with respect to T_{2n} is defined as the ratio of their asymptotic mean-square errors (AMSE)

$$\mathrm{ARE}_\mu(T_{1n}, T_{2n}) = \frac{\mathrm{AMSE}_\mu(T_{2n})}{\mathrm{AMSE}_\mu(T_{1n})}$$

where

$$\mathrm{AMSE}_\mu(T_{jn}) = \lim_{n\longrightarrow\infty} \mathrm{E}_{f_{T_{jn}|\mu}}[(T_{jn} - \tau(\mu))^2] \qquad j = 1, 2$$

Find $\mathrm{ARE}_\mu(T_{1n}, T_{2n})$.

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3 Suppose that $(X_i, Y_i), i = 1, \ldots, n$ are independent pairs of random variables with joint pdf

$$f_{X,Y|\theta,\phi}(x, y|\theta, \phi) = \phi^2\theta \exp\{-[\phi x + \theta\phi y]\} \qquad x, y > 0$$

for parameters $\theta, \eta > 0$. Find the maximum likelihood estimator, $\widehat{\underset{\sim}{\theta}}$, of $\underset{\sim}{\theta} = (\theta, \phi)^{\mathsf{T}}$, and a large sample approximation its distribution, given in this regular case by

$$\sqrt{n}(\widehat{\underset{\sim}{\theta}} - \underset{\sim}{\theta}) \xrightarrow{d} Z \sim \text{Normal}(\underset{\sim}{0}, \mathcal{I}(\underset{\sim}{\theta})^{-1})$$

where $\mathcal{I}(\underset{\sim}{\theta})$ is the Fisher Information.

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