556: MATHEMATICAL STATISTICS I

MULTIVARIATE PROBABILITY DISTRIBUTIONS

1 The Multinomial Distribution

The multinomial distribution is a multivariate generalization of the binomial distribution. The binomial distribution can be derived from an "infinite urn" model with two types of objects being sampled without replacement. Suppose that the proportion of "Type 1" objects in the urn is θ (so $0 \leq \theta \leq 1$) and hence the proportion of "Type 2" objects in the urn is $1-\theta$. Suppose that n objects are sampled, and X is the random variable corresponding to the number of "Type 1" objects in the sample. Then $X \sim Bin(n, \theta)$, and

$$f_X(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \qquad x \in \{0, 1, 2, ..., n\}$$

Now consider a generalization; suppose that the urn contains $k+1$ types of objects ($k = 1, 2, ...$), with θ_i being the proportion of Type i objects, for $i = 1, ..., k+1$. Let X_i be the random variable corresponding to the number of type i objects in a sample of size n, for $i = 1, ..., k$. Then the joint pmf of vector $\underset{\sim}{X} = (X_1, ..., X_k)^\mathsf{T}$ is given by

$$f_{X_1,...,X_k}(x_1, ..., x_k) = \frac{n!}{x_1!...x_k!x_{k+1}!} \theta_1^{x_1} ....\theta_k^{x_k} \theta_{k+1}^{x_{k+1}} = \frac{n!}{x_1!...x_k!x_{k+1}!} \prod_{i=1}^{k+1} \theta_i^{x_i}$$

where $0 \leq \theta_i \leq 1$ for all i, and $\theta_1 + ... + \theta_k + \theta_{k+1} = 1$, and where x_{k+1} is defined by

$$x_{k+1} = n - (x_1 + ... + x_k).$$

This is the mass function for the **multinomial distribution** which reduces to the binomial if $k = 1$.

2 The Dirichlet Distribution

The Dirichlet distribution is a multivariate generalization of the Beta distribution. The joint pdf of vector $\underset{\sim}{X} = (X_1, ..., X_k)^\mathsf{T}$ is given by

$$f_{X_1,...,X_k}(x_1, ..., x_k) = \frac{\Gamma(\alpha)}{\Gamma(\alpha_1)...\Gamma(\alpha_k)\Gamma(\alpha_{k+1})} x_1^{\alpha_1 - 1}...x_k^{\alpha_k - 1} {x_{k+1}}^{\alpha_{k+1}-1}$$

for $0 \leq x_i \leq 1$ for all i such that $x_1 + ... + x_k + x_{k+1} = 1$, where $\alpha = \alpha_1 + ... + \alpha_{k+1}$ and where x_{k+1} is defined by

$$x_{k+1} = 1 - (x_1 + ... + x_k).$$

This is the density function which reduces to the Beta distribution if $k = 1$. It can also be shown that the marginal distribution of X_i is $Beta(\alpha_i, \alpha)$.

3 The Multivariate Normal Distribution

The **multivariate normal distribution** is a multivariate generalization of the normal distribution. The joint pdf of $\underset{\sim}{X} = (X_1, ..., X_k)^\mathsf{T}$ takes the form

$$f_{X_1,...,X_k}(x_1, ..., x_k) = \left(\frac{1}{2\pi}\right)^{k/2} \frac{1}{|\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\underset{\sim}{x} - \underset{\sim}{\mu})^\mathsf{T} \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})\right\}$$

where $\underset{\sim}{x} = (x_1, ..., x_k)^\mathsf{T}$, $\underset{\sim}{\mu}$ is a $k \times 1$ vector, and Σ is a symmetric, positive-definite $k \times k$ matrix.

It can be shown that all marginal and all conditional distributions derived from the multivariate normal are also multivariate normal, and that any linear combination

$$\underset{\sim}{Y} = A\underset{\sim}{X}$$

for matrix A also has a multivariate normal distribution.