556: MATHEMATICAL STATISTICS I
COMPUTING THE HYPERBOLIC SECANT DISTRIBUTION CHARACTERISTIC FUNCTION

David A. Stephens
Department of Mathematics and Statistics
McGill University

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Abstract

We give two methods for computing the characteristic function of the hyperbolic secant (sech) distribution. The first utilizes Fourier series expansions, the second complex analysis.

1 Introduction

We aim to compute the characteristic function for pdf

$$f_X(x) = \frac{1}{\cosh(\pi x)} = \frac{2}{e^{-\pi x} + e^{\pi x}} = 2\sum_{k=0}^{\infty}(-1)^k \exp\{-(2k+1)\pi|x|\} \qquad x \in \mathbb{R}. \tag{1}$$

that corresponds to the *hyperbolic secant* distribution[1].

2 Using series expansions

2.1 A series expansion for the characteristic function

Note first that the expansion in equation (1) is generated as follows. Consider first $x > 0$; we have

$$\frac{2}{e^{-\pi x} + e^{\pi x}} = \frac{2e^{-\pi x}}{1 + e^{-2\pi x}} = 2e^{-\pi x}\sum_{k=0}^{\infty}(-1)^k \exp\{-2\pi k x\} = 2\sum_{k=0}^{\infty}(-1)^k \exp\{-(2k+1)\pi x\} \tag{2}$$

But for $x > 0$, $f_X(-x) = f_X(x)$, so equation (2) holds for $x < 0$ also with x replaced by $-x$. Thus

$$f_X(x) = 2\sum_{k=0}^{\infty}(-1)^k \exp\{-(2k+1)\pi|x|\} \qquad x \in \mathbb{R}.$$

Note that

$$\int_{-\infty}^{\infty} \frac{2e^{tx}}{e^{-\pi x} + e^{\pi x}}\, dx < \infty$$

for $|t| < \pi$, so the mgf exists. Thus,

$$\begin{aligned} C_X(t) &= 2\int_{-\infty}^{\infty} e^{itx} \sum_{k=0}^{\infty}(-1)^k \exp\{-(2k+1)\pi|x|\}\, dx = 2\sum_{k=0}^{\infty}(-1)^k \int_{-\infty}^{\infty} e^{itx} \exp\{-(2k+1)\pi|x|\}\, dx \\ &= 2\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)\pi} \frac{2}{1 + \left\{\dfrac{t}{(2k+1)\pi}\right\}^2} \\ &= 4\pi\sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)}{(2k+1)^2\pi^2 + t^2} \end{aligned} \tag{3}$$

using the result from lectures that

$$\int_{-\infty}^{\infty} e^{itx}\frac{1}{2}e^{-|x|}\, dx = \frac{1}{1+t^2}.$$

[1]The lead factor of 2 was inadvertently dropped in the previous calculations

2.2 A Fourier series expansion for sech(x)

A Fourier series expansion for function $f(x)$ takes the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx)) \tag{4}$$

where for $k \geq 0$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx.$$

Note that if $f(x)$ is an even function, then $f(x)\sin(kx)$ is odd and $f(x)\cos(kx)$ is even, so

$$a_k = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\ dx.$$

and $b_k = 0$. Conversely, if $f(x)$ is odd, then $f(x)\sin(kx)$ is even and $f(x)\cos(kx)$ is odd, so

$$b_k = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(kx)\ dx.$$

and $a_k = 0$.

- Consider first the expansion of $f(x) = \cos(\theta x)$, where θ is not integer valued. This is an even function, so we have $b_k = 0$ and

$$\begin{aligned}
\frac{\pi}{2}a_k &= \int_0^{\pi} f(x)\cos(kx)\ dx = \int_0^{\pi}\cos(\theta x)\cos(kx)\ dx = \frac{1}{2}\int_0^{\pi}\cos((\theta+k)x) + \cos((\theta-k)x)\ dx \\
&= \frac{1}{2}\left[\frac{\sin((\theta+k)x)}{(\theta+k)} + \frac{\sin((\theta-k)x)}{(\theta-k)}\right]_0^{\pi} \\
&= \frac{1}{2}\frac{\sin((\theta+k)\pi)}{(\theta+k)} + \frac{\sin((\theta-k)\pi)}{(\theta-k)} \\
&= \frac{1}{2}\frac{(\theta-k)\sin((\theta+k)\pi) + (\theta+k)\sin((\theta-k)\pi)}{(\theta+k)(\theta-k)} \\
&= \frac{1}{2}\frac{(\theta-k)\sin(\theta\pi)\cos(k\pi) + (\theta-k)\cos(\theta\pi)\sin(k\pi) + (\theta+k)\sin(\theta\pi)\cos(-k\pi) + (\theta+k)\cos(\theta\pi)\sin(-k\pi)}{(\theta^2-k^2)} \\
&= \frac{1}{2}\frac{(\theta-k)\sin(\theta\pi)(-1)^k + (\theta+k)\sin(\theta\pi)(-1)^{k+1}}{(\theta^2-k^2)} \\
&= (-1)^k\frac{\theta\sin(\theta\pi)}{(\theta^2-k^2)}
\end{aligned}$$

  Also,

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\cos(\theta x)\ dx = \frac{2\sin(\theta\pi)}{\theta\pi}.$$

  Hence, from equation (4)

$$\begin{aligned}
\cos(\theta x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) \\
&= \frac{\sin(\theta\pi)}{\theta\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{\theta\sin(\theta\pi)}{(\theta^2-k^2)}\cos(kx) \\
&= \frac{\sin(\theta\pi)}{\theta\pi} + \frac{2\theta\sin(\theta\pi)}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{1}{(\theta^2-k^2)}\cos(kx) \\
&= \frac{2\theta\sin(\theta\pi)}{\pi}\left[\frac{1}{2\theta^2} - \frac{\cos(x)}{(\theta^2-1^2)} + \frac{\cos(2x)}{(\theta^2-2^2)} - \cdots\right]
\end{aligned} \tag{5}$$

- Now consider the expansion of $f(x) = \sin(\theta x)$, where θ is not integer valued. This is an odd function, so $a_k = 0$ and by similar calculation

$$\begin{aligned}\frac{\pi}{2} b_k &= \int_0^\pi f(x)\sin(kx)\ dx = \int_0^\pi \sin(\theta x)\sin(kx)\ dx \\ &= -\frac{1}{2}\int_0^\pi \cos((\theta+k)x) - \cos((\theta-k)x)\ dx \\ &= (-1)^k \frac{k\sin(\theta\pi)}{(\theta^2 - k^2)}\end{aligned}$$

Hence, from equation (4)

$$\begin{aligned}\sin(\theta x) &= \sum_{k=1}^{\infty} b_k \sin(kx) = \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k \frac{k\sin(\theta\pi)}{(\theta^2-k^2)}\sin(kx) \\ &= \frac{2\sin(\theta\pi)}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{k\sin(kx)}{(\theta^2-k^2)} \\ &= -\frac{2\sin(\theta\pi)}{\pi}\left[\frac{\sin(x)}{(\theta^2-1^2)} - \frac{2\sin(2x)}{(\theta^2-2^2)} + \frac{3\sin(3x)}{(\theta^2-3^2)}\cdots\right] \quad (6)\end{aligned}$$

Now, for any θ,

$$\sin(\theta\pi) = 2\sin(\theta\pi/2)\cos(\theta\pi/2)$$

so that

$$\frac{1}{\cos(\theta\pi/2)} = \frac{2\sin(\theta\pi/2)}{\sin(\theta\pi)}. \quad (7)$$

But from equation (6), for any x

$$\frac{\sin(\theta x)}{\sin(\theta\pi)} = \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{k\sin(kx)}{(\theta^2-k^2)}$$

and at $x = \pi/2$,

$$\frac{\sin(\theta\pi/2)}{\sin(\theta\pi)} = \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{k\sin(k\pi/2)}{(\theta^2-k^2)} = \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)}{(\theta^2-(2k-1)^2)}$$

Hence, from equation (7)

$$\frac{1}{\cos(\theta\pi/2)} = \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)}{(\theta^2-(2k-1)^2)}$$

and equivalently, at $\theta = 2x$

$$\frac{\pi}{\cos(\pi x)} = 4\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)}{(4x^2-(2k-1)^2)}$$

or at $\theta = 2ix$

$$\frac{\pi}{\cosh(\pi x)} = 4\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)}{(-4x^2-(2k-1)^2)} = 4\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2k-1)}{((2k-1)^2+4x^2)}$$

and hence

$$\frac{1}{\cosh(\pi x)} = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{(2k+1)}{((2k+1)^2+4x^2)} \quad (8)$$

2.3 The Characteristic Function of the sech(x) distribution

Setting $x = t/2\pi$ in equation (8), we conclude that

$$\begin{aligned}\frac{1}{\cosh(t/2)} &= \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{(2k+1)}{((2k+1)^2+t^2/\pi^2)} \\ &= 4\pi\sum_{k=0}^{\infty}(-1)^k\frac{(2k+1)}{(2k+1)^2\pi^2+t^2}\end{aligned}$$

and

$$C_X(t) = \frac{1}{\cosh(t/2)} = \text{sech}(t/2) = \frac{2}{e^{t/2}+e^{-t/2}} \quad (9)$$

by comparison with equation (3).

2.4 A note on the Fourier series representation

Technically, the Fourier series expansion in equation (4) is valid on $|x| < \pi$, yet clearly the cf C_X is defined for all $t \in \mathbb{R}$ as

$$\left| \int_{-\infty}^{\infty} e^{itx} \frac{2}{e^{\pi x} + e^{-\pi x}} \, dx \right| < 1 \qquad \forall \, t.$$

The cf is not identically zero when $|t/2| < \pi$; in fact it does follow that the formula in equation (9) is valid for all t as the function sech is analytic at all values of t, and thus

$$C_X(t) = \text{sech}(t/2) \qquad t \in \mathbb{R}$$

by the technique of analytic continuation.

3 A Proof using Complex Analysis

We now compute the result using complex analysis. This proof is adapted from the proof of Priestley[2], p241. Consider the integral of the complex-valued function

$$f(z) = \frac{e^{az}}{\cosh(z)} = \frac{2e^{az}}{e^z + e^{-z}} \qquad z \in \mathbb{C}$$

for $a \in \mathbb{R}$, and $-1 < a < 1$; f has simple poles at

$$z = \frac{1}{2}(2k+1)\pi i \qquad k \in \mathbb{Z}$$

as

$$\exp\left\{\frac{1}{2}(2k+1)\pi i\right\} + \exp\left\{-\frac{1}{2}(2k+1)\pi i\right\} = 2\cos\left(\frac{1}{2}(2k+1)\pi\right) = 0.$$

Technically, f is *holomorphic* inside and on the rectangular contour C defined (anti-clockwise) by the corners

$$(R, 0), \ (R, \pi), \ (-S, \pi), \ (-S, 0),$$

except at the pole $z_0 = \pi i/2$. The **residue** at this *covert* pole is defined as

$$\text{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

where

$$g(z) = 2e^{az} \qquad h(z) = e^z + e^{-z} \ \therefore \ h'(z) = e^z - e^{-z}$$

so that

$$\text{Res}(f(z), z_0) = \frac{2e^{a\pi i/2}}{e^{\pi i/2} - e^{-\pi i/2}} = \frac{2e^{a\pi i/2}}{2i\sin(\pi/2)} = -ie^{a\pi i/2}$$

Then by Cauchy's Residue Theorem, the integral around the contour is equal to the residue multiplied by $2\pi i$, that is

$$\oint_C f(z)dz = 2\pi i \text{Res}(f(z), z_0) = 2\pi e^{a\pi i/2} \tag{10}$$

For the line integral

$$\begin{aligned}
\oint_C f(z)dz &= \int_0^{\pi} f(R+iy) \, dy + \int_R^{-S} f(x+i\pi) \, dx + \int_{\pi}^{0} f(-S+iy) \, dy + \int_{-S}^{R} f(x+i0) \, dx \\
&= \int_0^{\pi} \frac{2e^{a(R+iy)}}{e^{(R+iy)} + e^{-(R+iy)}} \, dy + \int_R^{-S} \frac{2e^{a(x+i\pi)}}{e^{(x+i\pi)} + e^{-(x+i\pi)}} \, dx \\
&\quad + \int_{\pi}^{0} \frac{2e^{a(-S+iy)}}{e^{(-S+iy)} + e^{-(-S+iy)}} \, dy + \int_{-S}^{R} \frac{2e^{a(x+i0)}}{e^{(x+i0)} + e^{-(x+i0)}} \, dx
\end{aligned}$$

Taking the integrals in turn, and examining limiting behaviour as the $R, S \to \infty$

$$\left| \int_0^{\pi} \frac{2e^{a(R+iy)}}{e^{(R+iy)} + e^{-(R+iy)}} \, dy \right| \leq \int_0^{\pi} \left| \frac{2e^{a(R+iy)}}{e^{(R+iy)} + e^{-(R+iy)}} \right| \, dy \leq \int_0^{\pi} \left| \frac{2e^{aR}}{e^R - e^{-R}} \right| \, dy \to 0$$

[2] *Introduction to Complex Analysis, 2nd Edition*, H. A. Priestley, 2003, Oxford University Press.

as $R \to \infty$, as $a < 1$. Similarly

$$\left| \int_0^\pi \frac{2e^{a(-S+iy)}}{e^{(-S+iy)} + e^{-(-S+iy)}} \; dy \right| \leq \int_0^\pi \left| \frac{2e^{a(-S+iy)}}{e^{(-S+iy)} + e^{-(-S+iy)}} \right| \; dy \leq \int_0^\pi \left| \frac{2e^{-aS}}{e^{-S} - e^{S}} \right| \; dy \to 0$$

as $S \to \infty$, as $a > -1$. For the remaining integrals

$$\int_R^{-S} \frac{2e^{a(x+i\pi)}}{e^{(x+i\pi)} + e^{-(x+i\pi)}} \; dx = -e^{a\pi i} \int_R^{-S} \frac{2e^{ax}}{e^x + e^{-x}} \; dx = e^{a\pi i} \int_{-S}^{R} \frac{2e^{ax}}{e^x + e^{-x}} \; dx$$

$$\int_{-S}^{R} \frac{2e^{a(x+i0)}}{e^{(x+i0)} + e^{-(x+i0)}} \; dx = \int_{-S}^{R} \frac{2e^{ax}}{e^x + e^{-x}} \; dx$$

so neither of these integrands depend on R or S. Thus

$$\lim_{R,S\to\infty} \oint_C f(z)dz = (1 + e^{a\pi i}) \int_{-\infty}^{\infty} \frac{2e^{ax}}{e^x + e^{-x}} \; dx$$

and hence, from equation (10), we have

$$(1 + e^{a\pi i}) \int_{-\infty}^{\infty} \frac{2e^{ax}}{e^x + e^{-x}} \; dx = 2\pi e^{a\pi i/2}$$

so that

$$\int_{-\infty}^{\infty} \frac{2e^{ax}}{e^x + e^{-x}} \; dx = \frac{2\pi e^{a\pi i/2}}{(1 + e^{a\pi i})} = \frac{2\pi}{e^{a\pi i/2} + e^{-a\pi i/2}} = \frac{\pi}{\cos(a\pi/2)} = \pi \sec(a\pi/2).$$

Making the change of variable $x \to \pi x$ yields

$$\int_{-\infty}^{\infty} \frac{2e^{a\pi x}}{e^{\pi x} + e^{-\pi x}} \; dx = \sec(a\pi/2)$$

and setting $t = a\pi$ yields

$$\int_{-\infty}^{\infty} \frac{2e^{tx}}{e^{\pi x} + e^{-\pi x}} \; dx = \sec(t/2).$$

Thus

$$M_X(t) = \int_{-\infty}^{\infty} \frac{2e^{tx}}{e^{\pi x} + e^{-\pi x}} \; dx = \sec(t/2)$$

and

$$C_X(t) = \sec(it/2) = \text{sech}(t/2) = \frac{1}{\cosh(t/2)}.$$