MATH 556 - PRACTICE QUESTIONS

1. The joint pdf for continuous random variables X and Y with ranges $\mathbb{X} \equiv \mathbb{Y} \equiv R^+$ is given by

$$f_{X,Y}(x,y) = c_1 (x + 2y) \exp\{-(x+y)\} \qquad x, y > 0$$

and zero otherwise for some constant c_1.

(a) Find the value of c_1

(b) Find the probability

$$P[Y > X]$$

2. The joint pdf for continuous random variables X and Y is given by

$$f_{X,Y}(x,y) = c_2 \exp\{-2x - y\} \qquad 0 < x < y < \infty$$

and zero otherwise for some constant c_2.

(a) Find the marginal pdfs of X and Y, f_X and f_Y.

(b) Are X and Y independent random variables ? Justify your answer.

3. Let $U \sim Uniform(0,1)$, and let random variable X be defined in terms of U by

$$X = \begin{cases} \dfrac{1}{\lambda} \log 2U & 0 < U \leq \dfrac{1}{2} \\[2ex] -\dfrac{1}{\lambda} \log(2 - 2U) & \dfrac{1}{2} < U < 1 \end{cases}$$

for some parameter $\lambda > 0$.

Show that the pdf for X, f_X, is symmetric about zero. Find the characteristic function of X,

$$C_X(t) = E[e^{itX}]$$

SOLUTIONS

1. (a) To compute c_1: need to integrate $f_{X,Y}$ over $\mathbb{R}^+ \times \mathbb{R}^+$

$$\begin{aligned}
\int_0^\infty \int_0^\infty f_{X,Y}(x,y)\, dx\, dy &= c_1 \int_0^\infty \int_0^\infty (x+2y)\, e^{-(x+y)}\, dx\, dy \\
&= c_1 \int_0^\infty \left\{ \int_0^\infty (x+2y)\, e^{-(x+y)}\, dx \right\} e^{-y}\, dy \\
&= c_1 \int_0^\infty \left\{ \left[-(x+2y)\, e^{-x} \right]_0^\infty + \int_0^\infty e^{-x}\, dx \right\} e^{-y}\, dy \\
&= c_1 \int_0^\infty \left\{ 2y + \left[e^{-x} \right]_0^\infty \right\} e^{-y}\, dy \\
&= c_1 \int_0^\infty (2y+1)\, e^{-y}\, dy \\
&= c_1 \left\{ \left[-(2y+1)\, e^{-y} \right]_0^\infty + \int_0^\infty e^{-y}\, dy \right\} \\
&= c_1 \{2+1\} = 3c_1
\end{aligned}$$

and therefore $c_1 = 1/3$.

(b)

$$P[Y > X] = \iint_A f_{X,Y}(x,y)\, dx\, dy \qquad\qquad A \equiv \{(x,y) : 0 < x < y < \infty\}$$

so therefore

$$\begin{aligned}
P[Y > X] &= \int_0^\infty \int_0^y f_{X,Y}(x,y)\, dx\, dy \\
&= \int_0^\infty \left\{ \int_0^y \frac{1}{3}(x+2y)\, e^{-(x+y)}\, dx \right\} dy \\
&= \frac{1}{3} \int_0^\infty \left\{ \left[-(x+2y)e^{-x} \right]_0^y + \int_0^y e^{-x} \right\} e^{-y}\, dy \\
&= \frac{1}{3} \int_0^\infty \left\{ 2y - 3ye^{-y} + \left(1 - e^{-y}\right) \right\} e^{-y}\, dy \\
&= \frac{1}{3} \int_0^\infty \left\{ (2y+1)e^{-y} - (3y+1)e^{-2y} \right\} dy \\
&= \frac{1}{3} \int_0^\infty \left\{ (2y+1)e^{-y} \right\} dy - \int_0^\infty \left\{ (3y+1)e^{-2y} \right\} dy \\
&= \frac{1}{3} \left\{ \left[-(2y+1)e^{-y} \right]_0^\infty + \int_0^\infty 2e^{-y}\, dy \right\} - \frac{1}{3} \left\{ \left[-\frac{1}{2}(3y+1)e^{-2y} \right]_0^\infty + \int_0^\infty \frac{3}{2} e^{-2y}\, dy \right\} \\
&= \frac{1}{3}\{1+3\} - \frac{1}{3}\left\{ \frac{1}{2} + \frac{3}{4} \right\} \\
&= 1 - \frac{1}{3} \times \frac{5}{4} = \frac{7}{12}
\end{aligned}$$

2. Joint density

$$f_{X,Y}(x,y) = c_2 \exp\{-2x - y\} \qquad 0 < x < y < \infty$$

(a) For $x > 0$

$$\begin{aligned} f_X(x) &= c_2 \int_0^\infty f_{X,Y}(x,y)\; dy = c_2 \int_x^\infty e^{-(2x+y)}\; dy \\ &= c_2 e^{-2x} \int_x^\infty e^{-y}\, dy = c_2 e^{-2x} \left[-e^{-x}\right]_x^\infty = c_2 e^{-3x} \end{aligned}$$

But

$$\int_0^\infty c_2 e^{-3x}\, dx = c_2 \left[-\frac{e^{-3x}}{3}\right]_0^\infty = \frac{c_2}{3} \qquad \therefore \qquad c_2 = 3$$

and hence

$$f_X(x) = 3e^{-3x} \qquad x > 0$$

so that $X \sim Exponential(3)$.

Similarly for $y > 0$

$$\begin{aligned} f_Y(y) &= c_2 \int_0^\infty f_{X,Y}(x,y)\; dx = c_2 \int_0^y e^{-(2x+y)}\; dx \\ &= c_2 e^{-y} \int_0^y e^{-2x}\, dx \\ &= c_2 e^{-y} \left[-\frac{e^{-2x}}{2}\right]_0^y = \frac{c_2}{2} e^{-y}\left(1 - e^{-2y}\right) \end{aligned}$$

and hence

$$f_Y(y) = \frac{3}{2} e^{-y}\left(1 - e^{-2y}\right) \qquad y > 0$$

(b) X and Y are **not independent**.

This can be deduced in many ways; for example, if we take a particular point $(x,y) \in \mathbb{R}^2$ such that $x > y$. At this point, $f_{X,Y}(x,y) = 0$, but $f_X(x) > 0$ and $f_Y(y) > 0$ so

$$f_{X,Y}(x,y) \neq f_X(x) f_Y(y).$$

3. We have $U \sim Uniform(0,1)$ and

$$X = \begin{cases} \dfrac{1}{\lambda}\log 2U & 0 < U \leq \dfrac{1}{2} \\ \\ -\dfrac{1}{\lambda}\log(2-2U) & \dfrac{1}{2} < U < 1 \end{cases}$$

for $\lambda > 0$. For $x \leq 0$ $\left(\text{so that } 0 < u \leq \frac{1}{2}\right)$

$$F_X(x) = P[X \leq x] = P\left[\frac{1}{\lambda}\log 2U \leq x\right] = P\left[U \leq \frac{1}{2}e^{\lambda x}\right] = \frac{1}{2}e^{\lambda x}$$

and for $x > 0$ $\left(\text{so that } \frac{1}{2} < u < 1\right)$

$$\begin{aligned} F_X(x) &= P[X \leq x] = P[X \leq 0] + P[0 < X \leq x] \\ &= \frac{1}{2} + P\left[0 < -\frac{1}{\lambda}\log(2-2U) \leq x\right] \\ &= \frac{1}{2} + P\left[\frac{1}{2}e^{-\lambda x} \leq 1 - U < \frac{1}{2}\right] \\ &= \frac{1}{2} + P\left[\frac{1}{2} < U \leq 1 - \frac{1}{2}e^{-\lambda x}\right] \\ &= \frac{1}{2} + 1 - \frac{1}{2}e^{-\lambda x} - \frac{1}{2} = 1 - \frac{1}{2}e^{-\lambda x} \end{aligned}$$

Hence, by differentiation

$$f_X(x) = \begin{cases} \dfrac{\lambda}{2}e^{\lambda x} & 0 \leq x \\ \\ \dfrac{\lambda}{2}e^{-\lambda x} & x > 0 \end{cases} = \frac{\lambda}{2}e^{-\lambda|x|} \qquad x \in \mathbb{R}$$

which is symmetric about zero.

For the cf; by inspection, we see that the mgf exists, so we compute that instead (as the calculation is slightly easier)

$$\begin{aligned} E_{f_X}\left[e^{tX}\right] &= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{\infty} e^{tx}\frac{\lambda}{2}e^{-\lambda|x|}\,dx \\ &= \frac{\lambda}{2}\left[\int_{-\infty}^{0} e^{(\lambda+t)x}\,dx + \int_{0}^{\infty} e^{-(\lambda-t)x}\,dx\right] \\ &= \frac{\lambda}{2}\left[\frac{1}{\lambda+t} + \frac{1}{\lambda-t}\right] \qquad \text{if } |t| < \lambda \\ &= \frac{\lambda^2}{(\lambda+t)(\lambda-t)} = \frac{\lambda^2}{(\lambda^2-t^2)} = \frac{1}{1-\dfrac{t^2}{\lambda^2}} \end{aligned}$$

Hence the characteristic function is

$$C_X(t) = M_X(it) = \frac{\lambda^2}{\lambda^2-(it)^2} = \frac{\lambda^2}{\lambda^2+t^2} \qquad t \in \mathbb{R}$$