MATH 556 - MID-TERM EXAMINATION 2007

Marks can be obtained by answering all questions. All questions carry equal marks.

1. (a) Discrete random variable X has pmf f_X given by

$$f_X(x) = \frac{-1}{\log(1-\phi)} \frac{\phi^x}{x} \qquad x = 1, 2, 3, \ldots$$

and zero otherwise, for parameter ϕ, where $0 < \phi < 1$.

(i) Find the moment generating function (mgf) for X, $M_X(t)$.

(ii) Find the expectation of X.

(b) Continuous random variable X has pdf f_X given by

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \theta x & 0 \leq x < 1 \\ \theta e^{-2(x-1)} & x \geq 1 \end{cases}$$

for some parameter θ.

Find the value of θ, and the corresponding cdf, F_X.

2. In answering this question, you may quote results from the formula sheet.

(a) The joint pmf or pdf of random variables X and Y can be specified in the following way:

$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y).$$

(i) Find the marginal pmf of X if

$$X|Y = y \sim Binomial(n, y)$$

for positive integer n, and continuous random variable Y has a $Uniform(0, 1)$ marginal distribution.

(ii) Find the marginal pdf of X if

$$X|Y = y \sim Exponential(y)$$

and $Y \sim Exponential(\beta)$ for parameter $\beta > 0$.

(b) Suppose that random variables $Z_1, \ldots, Z_n$ are independent and identically distributed, having a $Normal(0, 1)$ distribution.

(i) Identify the distribution of $Y_1 = Z_1^2$. **Show your working.**

(ii) Identify the distribution of S_n where

$$S_n = \sum_{i=1}^{n} Z_i^2.$$

Justify your answer.

3. (a) Suppose that U_1 and U_2 are independent and identically distributed random variables having a $Uniform(0, 1)$ distribution. By direct consideration of the cdf, or otherwise, find the pdf of random variable

$$X = U_1 U_2.$$

Hint: consider an appropriate region of $[0, 1] \times [0, 1]$*, and the ranges of integration carefully.*

(b) If X_1 and X_2 are independent random variables, find the covariance between random variables

$$Y_1 = X_1 + X_2 \qquad\qquad Y_2 = X_1 - X_2$$

Are Y_1 and Y_2 independent ? Justify your answer.

Recall that the covariance between Y_1 *and* Y_2 *is given by*

$$Cov_{f_{Y_1}, f_{Y_2}}[Y_1, Y_2] = E_{f_{Y_1}, f_{Y_2}}[(Y_1 - \mu_1)(Y_2 - \mu_2)]$$

where μ_1 *and* μ_2 *are the expectations of* Y_1 *and* Y_2 *respectively.*

4. Suppose that $C_1(t)$ and $C_2(t)$ are the characteristic functions for continuous random variables with pdfs f_1 and f_2, and that $0 < \alpha < 1$.

(a) Show that

$$C_X(t) = \alpha C_1(t) + (1 - \alpha) C_2(t)$$

is the characteristic function for a continuous random variable, X say, and find the corresponding pdf of X, f_X.

(b) Find the characteristic function of random variable $Y = -3X + 2$.

(c) Identify continuous random variables Z_1 and Z_2 with characteristic functions

$$C_{Z_1}(t) = \{C_X(t)\}^2 \qquad\qquad C_{Z_2}(t) = |C_X(t)|^2$$

where $|C_X(t)|$ is the modulus of the complex-valued function $C_X(t)$.

Hint: for any complex number $z = re^{i\omega}$*, say,*

$$|z|^2 = z\overline{z}$$

where $\overline{z} = re^{-i\omega}$ *is the complex conjugate of* z*.*

(d) Find the distribution (pmf, pdf or cdf) of random variable Y with the following characteristic function:

$$C_Y(t) = \frac{1 + e^{2it} + e^{-2it}}{3} \qquad t \in \mathbb{R}$$