MATH 556 - ASSIGNMENT 4
SOLUTIONS

1 By properties of the Gamma distribution, we can write

$$V_n = \frac{(n-1)s_n^2}{\sigma^2} \sim \chi^2_{n-1} \equiv Gamma\left(\frac{n-1}{2}, \frac{1}{2}\right) \Longrightarrow V_n \stackrel{d}{=} \sum_{i=1}^{n-1} X_i$$

where $X_i \sim \chi^2_1 \equiv Gamma\,(1/2, 1/2)$, and the symbol $\stackrel{d}{=}$ indicates equality in distribution (that is, the left hand side and the right hand side have the same distribution). Using the Gamma expectation and variance results

$$\mathrm{E}_{f_{X_i}}[X_i] = \frac{1/2}{1/2} = 1 = \mu_X \qquad \mathrm{Var}_{f_{X_i}}[X_i] = \frac{1/2}{(1/2)^2} = 2 = \sigma^2_X$$

say. Hence, by the Central Limit Theorem,

$$\frac{V_n - (n-1)\mu_X}{\sqrt{(n-1)\sigma^2_X}} = \frac{V_n - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} Z \sim N(0,1)$$

Hence, substituting in the definition for V_n,

$$\frac{\dfrac{(n-1)s_n^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} = \frac{\sqrt{n-1}\,(s_n^2 - \sigma^2)}{\sigma^2\sqrt{2}} \xrightarrow{d} Z \sim N(0,1)$$

and finally, by a location/scale transformation to $Z_n = \sigma^2 + \dfrac{\sigma^2\sqrt{2}}{\sqrt{n-1}}Z$, we have

$$s_n^2 \sim AN\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$$

Result used: C.

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2 Using the Central Limit Theorem, as

$$\mathrm{E}_{f_{X_i}}[X_i] = \lambda \qquad \mathrm{Var}_{f_{X_i}}[X_i] = \lambda,$$

we may deduce directly that

$$\sqrt{n}(T_n - \lambda) \xrightarrow{d} Z \sim N(0, \lambda)$$

Now if $g(x) = e^{-x}$, then $\dot{g}(x) = -e^{-x}$, and by the Delta Method

$$\sqrt{n}(Y_n - e^{-\lambda}) \xrightarrow{d} Z \sim N(0, \lambda e^{-2\lambda})$$

and hence, if n is large.

$$Y_n \sim AN(e^{-\lambda}, \lambda e^{-2\lambda}/n)$$

Thus $\mu_n = e^{-\lambda}$ and $\sigma_n^2 = \lambda e^{-2\lambda}/n$.

Results used: C and D. 5 MARKS

3 (i) Using a similar argument to the one covered in lectures relating to order statistics, $Y_n(x)$ takes values on the set $\{0, 1, \ldots, n\}$, and as the X_i are independent draws from F_X, so we can write

$$Y_n(x) \stackrel{d}{=} \sum_{i=1}^{n} I_{(-\infty,x]}(X_i)$$

where the $I_{(-\infty,x]}(X_1), \ldots, I_{(-\infty,x]}(X_n)$ are indicator random variables with

$$\Pr[I_{(-\infty,x]}(X_i) = 1] = \Pr[X_i \leq x] = F_X(x) \qquad \Pr[I_{(-\infty,x]}(X_i) = 0] = \Pr[X_i > x] = 1 - F_X(x)$$

Hence, we can deduce that $Y_n(x)$ is the sum of independent and identically distributed $Bernoulli(F_X(x))$ random variables, so that

$$Y_n(x) \sim Binomial(n, F_X(x))$$

and

$$E_{f_{Y_n(x)}}[Y_n(x)] = nF_X(x) \qquad Var_{f_{Y_n(x)}}[Y_n(x)] = nF_X(x)(1 - F_X(x))$$

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(ii) By the Weak Law of Large Numbers, we may deduce that

$$T_n \xrightarrow{p} F_X(x)$$

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Result used: B.

4 If the relevant expectation and variance are finite (these are the regularity conditions), by the Weak Law of Large Numbers, we may deduce that the statistic

$$L_n(\theta, \phi) = \frac{1}{n} \sum_{i=1}^{n} \log \left\{ \frac{f_X(X_i; \theta)}{f_X(X_i; \phi)} \right\} \xrightarrow{p} E_{f_{X|\theta}} \left[\log \left\{ \frac{f_X(X; \theta)}{f_X(X; \phi)} \right\} \right].$$

But from first principles

$$E_{f_{X|\theta}} \left[\log \left\{ \frac{f_X(x; \theta)}{f_X(x; \phi)} \right\} \right] = \int \log \left\{ \frac{f_X(x; \theta)}{f_X(x; \phi)} \right\} f_X(x; \theta) \, dx = K(\theta, \phi)$$

where $K(\theta, \phi)$ is the Kullback-Leibler divergence between the densities with different parameter values θ and ϕ.

Note that by properties of the KL divergence,

$$L_n(\theta, \phi) \xrightarrow{p} 0$$

if $\theta = \phi$ or if the densities at θ and ϕ are identical everywhere; in all other cases

$$L_n(\theta, \phi) \xrightarrow{p} \Delta > 0$$

say.

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Result used: B.