# MATH 556 - ASSIGNMENT 2
## SOLUTIONS

1.

(i) As

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2|X_1}(x_2|x_1)$$

we have

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \frac{1}{\sqrt{16\pi^2}}\exp\left\{-\frac{1}{8}\left[4x_1^2+(x_2-x_1)^2\right]\right\} = \frac{1}{\sqrt{16\pi^2}}\exp\left\{-\frac{1}{8}\left[5x_1^2-2x_1x_2+x_2^2\right]\right\} \\ &= \frac{1}{\sqrt{16\pi^2}}\exp\left\{-\frac{1}{2}\,\underset{\sim}{x}^\mathsf{T} P \underset{\sim}{x}\right\} \end{aligned}$$

where

$$P = \frac{1}{4}\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}.$$

Thus if we let $\underset{\sim}{\mu} = [0,0]^\mathsf{T}$, and

$$\Sigma = P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

by comparison with the Multivariate Distributions handout, we see that $\underset{\sim}{X} \sim Normal(\underset{\sim}{\mu},\Sigma)$ in $k=2$ dimensions. Note that $|\Sigma| = 4$, so the lead term

$$\frac{1}{\sqrt{16\pi^2}} = \left(\frac{1}{2\pi}\right)^{k/2}\frac{1}{|\Sigma|^{1/2}}$$

is correctly matched.

4 MARKS

(ii) To get the marginal distribution, we complete the square in $x_1$ in the exponent of the integrand:

$$\begin{aligned} f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_1 &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{16\pi^2}}\exp\left\{-\frac{1}{8}\left[4x_1^2+(x_2-x_1)^2\right]\right\}\,dx_1 \\ &= \frac{1}{\sqrt{16\pi^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{8}\left[5\left(x_1-\frac{x_2}{5}\right)^2+\frac{4}{5}x_2^2\right]\right\}\,dx_1 \\ &= \frac{1}{\sqrt{16\pi^2}}\exp\left\{-\frac{1}{8}\frac{4}{5}x_2^2\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{5}{8}\left(x_1-\frac{x_2}{5}\right)^2\right\}\,dx_1 \\ &= \frac{1}{\sqrt{16\pi^2}}\exp\left\{-\frac{1}{10}x_2^2\right\}\times\sqrt{2\pi(4/5)} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}x_2^2\right\} \end{aligned}$$

where $\sigma^2 = 5$. Hence, $X_2 \sim N(0,\sigma^2)$. In this proof, we have used the completing the square formula

$$A(x-a)^2 + B(x-b)^2 = (A+B)\left(x-\frac{Aa+Bb}{A+B}\right)^2 + \frac{AB}{A+B}(a-b)^2$$

and the fact that the integral in line 3 has an integrand proportional to the $Normal(x_2/5, 4/5)$ density.

4 MARKS

2. We seek first the conditional distribution of $T_1$ and $T_2$, given that $R = \sqrt{T_1^2 + T_2^2} \leq 1$. The restriction

$$\sqrt{T_1^2 + T_2^2} \leq 1$$

corresponds to $(T_1, T_2)$ lying within the unit circle centered at the origin. That is, conditional on $R \leq 1$, $(T_1, T_2) \in \mathcal{D}$ (the unit disc), we must have that the joint pdf of $(T_1, T_2)$ is proportional to the joint pdf on $(-1,1) \times (-1,1)$, so that conditional on $R \leq 1$, $f_{T_1,T_2}(t_1, t_2) \propto 1$ for $(t_1, t_2) \in \mathcal{D}$. By elementary area calculation, it follows that under the restriction,

$$f_{T_1,T_2}(t_1, t_2) = \frac{1}{\pi} \qquad (t_1, t_2) \in \mathcal{D}.$$

To parameterize points on $\mathcal{D}$, we consider the transformation from $(T_1, T_2)$ to $(R, S)$ where $S = \arctan(Y/X)$; $S$ is a random variable with range $(-\pi, \pi)$, and we have

$$T_1 = R\cos(S) \qquad T_2 = R\sin(S).$$

Thus we have by the multivariate transformation method

$$f_{R,S}(r, s) = f_{T_1,T_2}(r\cos(s), r\sin(s))|J(r, s)| \qquad 0 < r < 1, -\pi < s < \pi$$

The Jacobian $|J(r, s)|$ is

$$\left|\left[\begin{array}{cc} \dfrac{\partial t_1}{\partial r} & \dfrac{\partial t_1}{\partial s} \\ \dfrac{\partial t_2}{\partial r} & \dfrac{\partial t_2}{\partial s} \end{array}\right]\right| = \left|\left[\begin{array}{cc} \cos(s) & -r\cos(s) \\ \sin(s) & r\cos(s) \end{array}\right]\right| = r$$

$$f_{R,S}(r, s) = \frac{r}{\pi} = (2r) \times \frac{1}{2\pi} = f_R(r) f_S(s) \qquad 0 < r < 1, -\pi < s < \pi$$

Thus, in fact, $R$ and $S$ are **independent**. Now consider $X$ and $Y$. We have that

$$X = \frac{T_1}{R}\sqrt{-2\log R^2} = \cos(S)\sqrt{-2\log R^2} \qquad\qquad Y = \frac{T_2}{R}\sqrt{-2\log R^2} = \sin(S)\sqrt{-2\log R^2}$$

Note that this is a 1-1 transformation for $0 < R < 1, -\pi < S < \pi$. Inverting this transformation is straightforward; we have

$$R = \exp\{-(X^2 + Y^2)/4\} \qquad S = \arctan(Y/X).$$

Thus we have by the multivariate transformation method

$$f_{X,Y}(x, y) = f_{R,S}(\exp\{-(x^2 + y^2)/4\}, \arctan(y/x))|J(x, y)| \qquad (x, y) \in \mathbb{R}^2$$

The Jacobian $|J(x, y)|$ is

$$\left|\left[\begin{array}{cc} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \end{array}\right]\right| = \left|\left[\begin{array}{cc} -x\exp\{-(x^2+y^2)/4\}/2 & -y\exp\{-(x^2+y^2)/4\}/2 \\ -y/(x^2+y^2) & x/(x^2+y^2) \end{array}\right]\right| = \exp\{-(x^2+y^2)/4\}/2$$

and hence

$$f_{X,Y}(x, y) = \frac{\exp\{-(x^2+y^2)/4\}}{\pi}\exp\{-(x^2+y^2)/4\}/2 = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(x^2+y^2)\right\} \qquad (x, y) \in \mathbb{R}^2$$

from which we see that $X$ and $Y$ are independent, and distributed as $Normal(0, 1)$ variables.

8 MARKS

3. Note that, for $r = 2, 3, \ldots$, using the binomial expansion

$$m'_r = E_{f_X}[(X - m_1)^r] = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} E_{f_X}[X^j] m_1^{r-j} = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} m_j m_1^{r-j}$$

so that

$$\begin{aligned} m'_2 &= m_2 - m_1^2 \\ m'_3 &= m_3 - 3m_1 m_2 + 2m_1^3 \\ m'_4 &= m_4 - 4m_1 m_3 + 6m_1^2 m_2 - 3m_1^4 \end{aligned}$$

From first principles, for a Poisson random variable

$$E_{f_X}[X] = \sum_{x=0}^{\infty} x f_X(x) = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{(x-1)}}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Now, by the linearity of expectations

$$E_{f_X}[X^2] = E_{f_X}[X^2 - X + X] = E_{f_X}[X(X-1) + X] = E_{f_X}[X(X-1)] + E_{f_X}[X] = E_{f_X}[X(X-1)] + \lambda$$

and

$$E_{f_X}[X(X-1)] = \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!} = \lambda^2 e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \lambda^2$$

so that

$$E_{f_X}[X^2] = \lambda^2 + \lambda.$$

Similarly,

$$E_{f_X}[X^3] = E_{f_X}[X(X-1)(X-2)] + 3E_{f_X}[X^2] - 2E_{f_X}[X] = E_{f_X}[X(X-1)(X-2)] + 3\lambda^2 + 3\lambda - 2\lambda$$

Using similar arguments to above, we have $E_{f_X}[X(X-1)(X-2)] = \lambda^3$, so

$$E_{f_X}[X^3] = \lambda^3 + 3\lambda^2 + \lambda$$

Finally,

$$\begin{aligned} E_{f_X}[X^4] &= E_{f_X}[X(X-1)(X-2)(X-3)] + 6E_{f_X}[X^3] - 11E_{f_X}[X^2] + 6E_{f_X}[X] \\ &= E_{f_X}[X(X-1)(X-2)(X-3)] + 6(\lambda^3 + 3\lambda^2 + \lambda) - 11(\lambda^2 + \lambda) + 6\lambda \\ &= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda \end{aligned}$$

Bringing all these results together, we find that

$$m'_2 = \lambda \qquad m'_3 = \lambda \qquad m'_4 = 3\lambda^2 + \lambda$$

and hence

$$\text{skw}_{f_X}[X] = \frac{m'_3}{\{m'_2\}^{3/2}} = \lambda^{-1/2} \qquad \text{kur}_{f_X}[X] = \frac{m'_4}{\{m'_2\}^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \frac{1}{\lambda}$$

4 MARKS

*Note: Calculation using generating functions (mgf, fmgf, cgf) also possible.*