MATH 556 - ASSIGNMENT 1
SOLUTIONS

1. We have, for $x = 1, 2, \ldots$

$$F_X(x) = \sum_{t=1}^{x} f_X(t) = \sum_{t=1}^{x} \frac{k}{t(t+1)}$$

but

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

so, in fact

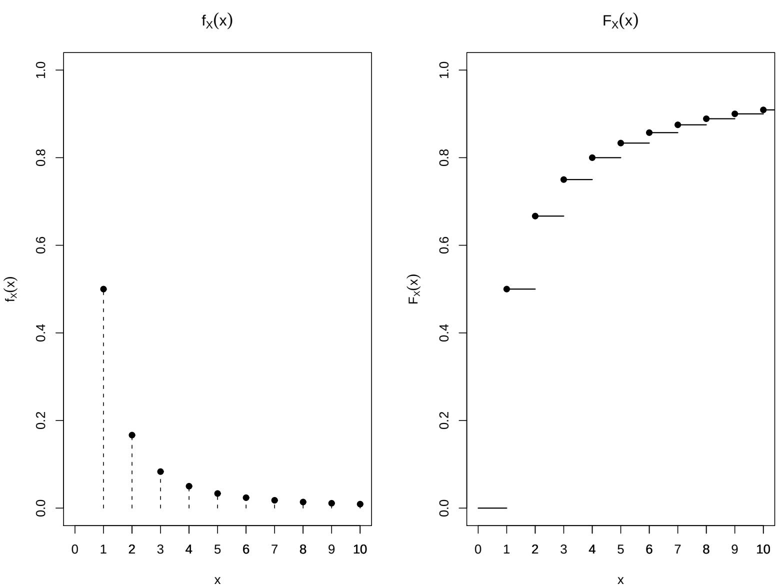
$$F_X(x) = k \sum_{t=1}^{x} \left[\frac{1}{t} - \frac{1}{t+1} \right] = k - \frac{k}{x+1} = \frac{kx}{x+1}$$

as the sum telescopes. Noting that we must have $F_X(x) \longrightarrow 1$ as $x \longrightarrow \infty$, it follows that $k = 1$.

Denoting by $\lfloor x \rfloor$ the largest integer not greater than x, we have that

$$F_X(x) = \frac{\lfloor x \rfloor}{\lfloor x \rfloor + 1} \qquad x \geq 0$$

and zero otherwise. See the sketches below:



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2. We note that for $x = 1, 2, \ldots$,

$$h_X(x) = \frac{f_X(x)}{1 - F_X(x-1)} = \frac{\Pr[X = x]}{1 - \Pr[X \leq x-1]} = \frac{\Pr[X = x]}{\Pr[X \geq x]}.$$

From the definition of conditional probability, we can identify that in this discrete setting

$$h_X(x) = \Pr[X = x | X \geq x]$$

as $(X = x) \cap (X \geq x) \equiv (X = x)$. Clearly, as h_X is a conditional probability, we must have

$$0 \leq h_X(x) \leq 1.$$

To find an f_X with a constant hazard, consider in turn $x = 1, 2, \ldots$. For $x = 1$,

$$h_X(1) = \frac{\Pr[X = 1]}{\Pr[X \geq 1]} = \Pr[X = 1] = \theta$$

say, for some θ with $0 \leq \theta \leq 1$. For $x = 2$,

$$h_X(2) = \frac{\Pr[X = 2]}{1 - \Pr[X \leq 1]} = \frac{\Pr[X = 2]}{1 - \theta}.$$

But we require that $h_X(2) = h_X(1) = \theta$, so therefore

$$\Pr[X = 2] = (1 - \theta)\theta.$$

By using this argument recursively, we see after some algebra that

$$f_X(x) = (1 - \theta)^{x-1}\theta \qquad x = 1, 2, \ldots$$

and zero otherwise.

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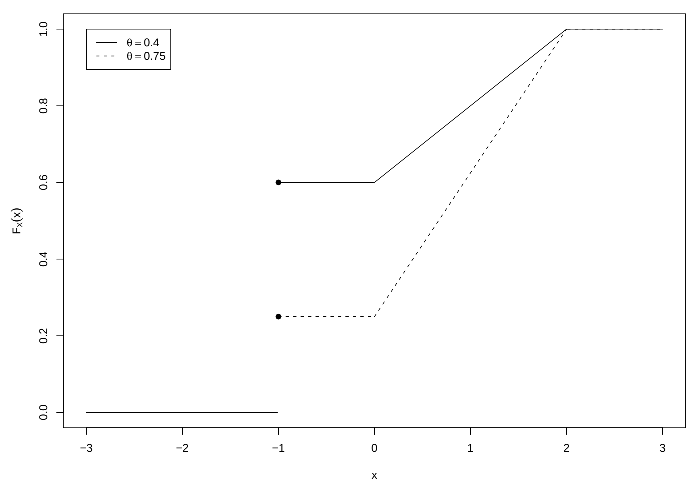
3. The plot below, F_X for two values of θ are shown.



(i) By definition

$$\Pr[X = -1] = \Pr[X \leq x] - \Pr[X < x]$$

so that

$$\Pr[X = -1] = F_X(-1) - \lim_{x \to -1^-} F_X(x) = (1 - \theta) - \lim_{x \to -1^-} 0 = 1 - \theta$$

where $x \to -1^-$ indicates that we take the limit as x tends to -1 from below.

(ii) As F_X is continuous at $x = 0$, we have $\Pr[X = 0] = 0$.

(iii) By the probability axioms

$$\Pr[X \geq 1] = 1 - P[X < 1] = 1 - (1 - \theta + \theta \times (1/2)) = \theta/2.$$

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4. We merely need to check that $F(y)$ has the properties of a cdf. Recall that the function $\sin(x)$ is a monotone increasing function for $0 < x < \pi/2$, with

$$\sin(0) = 0 \qquad \sin(\pi/2) = 1.$$

Now, by definition

$$\Pr[\sin(X) \leq y] = \int_{A_y} f_X(x)\, dx$$

where $A_y \equiv \{x : \sin(x) \leq y\}$. But

$$\sin(x) \leq y \quad \Longleftrightarrow \quad x \leq \arcsin(y)$$

so

$$\Pr[\sin(X) \leq y] = \Pr[X \leq \arcsin(y)] = \int_0^{\arcsin(y)} f_X(x)\, dx$$

and hence

$$F(y) = \frac{2}{\pi} \arcsin(y).$$

From here it is easy to verify that

- $F(0) = 0, F(1) = 1$
- F is non-decreasing
- F is continuous

By elementary calculus, the corresponding pdf is

$$f(y) = \frac{d}{dt}\left\{\frac{2}{\pi}\arcsin(t)\right\}_{t=y} = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}} \qquad 0 < y < 1$$

and zero otherwise.

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