MATH 556 - ASSIGNMENT 4

To be handed in not later than 5pm, 29th November 2007.
Please hand in during lectures, to Burnside 1235, or to the Mathematics Office Burnside 1005

In the following questions, use the key stochastic convergence concepts for a sequence of random variables $\{X_n : n \geq 1\} = \{X_1, X_2, \ldots\}$ with corresponding distribution functions $\{F_{X_n}, n \geq 1\}$.

A. **Convergence in Distribution :** Suppose that there exists a cdf, F_X, such that for all x at which F_X is continuous,

$$\lim_{n \longrightarrow \infty} F_{X_n}(x) = F_X(x).$$

Then $\{X_n\}$ **converges in distribution** to random variable X with cdf F_X, $X_n \stackrel{d}{\longrightarrow} X$, and F_X is the **limiting distribution**. An equivalent result holds for the convergence of mgfs.

B. **Convergence in Probability :** $\{X_n\}$ **converges in probability** to random variable X, $X_n \stackrel{p}{\longrightarrow} X$, if, for all $\epsilon > 0$,

$$\lim_{n \longrightarrow \infty} \Pr\left[|X_n - X| < \epsilon\right] = 1 \qquad \text{or} \qquad \lim_{n \longrightarrow \infty} \Pr\left[|X_n - X| \geq \epsilon\right] = 0$$

In particular, if $\{X_n\}$ are an i.i.d. sequence with finite expectation μ and variance σ^2, then the **Weak Law of Large Numbers** says that

$$\overline{X}_n \stackrel{p}{\longrightarrow} \mu \qquad \text{as} \qquad n \longrightarrow \infty.$$

C. **Central Limit Theorem :** Suppose $X_1, ..., X_n$ are i.i.d. random variables with finite expectation μ and variance σ^2. Let the random variable Z_n be defined by

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$$

where $\overline{X}_n$ is the sample mean random variable derived from $X_1, \ldots, X_n$. Then

$$Z_n \stackrel{d}{\longrightarrow} Z \sim N(0,1) \qquad \text{as} \qquad n \longrightarrow \infty.$$

We may deduce the **asymptotic Normal distribution** of $\overline{X}_n$ and write, for large finite n that

$$\overline{X}_n \sim AN(\mu, \sigma^2/n).$$

D. **The Delta Method :** Suppose that $\sqrt{n}(X_n - c) \stackrel{d}{\longrightarrow} X$ as $n \longrightarrow \infty$ for some constant c, and let $g(x)$ be a real-valued function such that $\dot{g}(c) \neq 0$, where $\dot{g}(x)$ is the first derivative of $g(x)$. Then

$$\sqrt{n}(g(X_n) - g(c)) \stackrel{d}{\longrightarrow} \dot{g}(c)X \qquad \text{as} \quad n \longrightarrow \infty.$$

In particular, if $X \sim N(0, \sigma^2)$, then

$$\sqrt{n}(g(X_n) - g(c)) \stackrel{d}{\longrightarrow} N(0, \{\dot{g}(c)\}^2\sigma^2) \qquad \text{as} \quad n \longrightarrow \infty.$$

so that, for large finite n

$$g(X_n) \sim AN(g(c), \{\dot{g}(c)\}^2\sigma^2/n).$$

Please Turn Over for questions 1,2,3,4.

In each of the following questions, state which of the results or definitions A, B, C or D is used to derive the answers.

1. Let s_n^2 denote the sample variance derived from a random sample of size n from a $Normal(\mu, \sigma^2)$ distribution. Show that

$$\frac{\sqrt{n-1}(s_n^2 - \sigma^2)}{\sigma^2\sqrt{2}} \xrightarrow{d} Z \sim N(0,1) \qquad \text{so that} \qquad s_n^2 \sim AN\left(\sigma^2, \frac{2\sigma^4}{n-1}\right)$$

where AN denotes an asymptotic normal approximation to the distribution of the random variable.

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2. Suppose that $X_1, \ldots, X_n$ are a random sample from a $Poisson(\lambda)$ distribution. Suppose that $T_n = \overline{X}_n$ is the corresponding sample mean random variable, and let $Y_n = e^{-T_n}$.

Show that, for large n, Y_n is approximately Normally distributed with parameters μ_n and σ_n^2 to be identified.

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3. Suppose that random variables $X_1, X_2, \ldots, X_n$ are a random sample from a probability distribution described by cdf F_X.

Let $Y_n(x)$ be the discrete random variable defined as the number (out of n) of the Xs that are **no greater than** x, for fixed $x \in \mathbb{R}$.

(i) Find the probability distribution of $Y_n(x)$, and state the expectation and variance of $Y_n(x)$.

Hint: consider the events "$X_i \leq x$" for $i = 1, \ldots, n$, and how they define $Y_n(x)$.

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(ii) State, in precise mathematical terms, the behaviour of the random variable $T_n(x)$,

$$T_n(x) = \frac{Y_n(x)}{n}$$

in the limit as $n \longrightarrow \infty$.

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4. Suppose that random variables $X_1, \ldots, X_n$ are a random sample from a probability distribution described by pdf $f_X(x|\theta)$ with parameter θ. Let ϕ be another value in the parameter space Θ, and let

$$L_n(\theta, \phi) = \boxed{\frac{1}{n}} \sum_{i=1}^{n} \log\left\{\frac{f_X(X_i;\theta)}{f_X(X_i;\phi)}\right\}$$

be a statistic derived from $X_1, \ldots, X_n$.

Show that $L_n(\theta, \phi)$ converges in probability to some constant to be identified, under regularity conditions to be stated.

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