MATH 556 - ASSIGNMENT 3

To be handed in not later than 5pm, 15th November 2007.
Please hand in during lectures, to Burnside 1235, or to the Mathematics Office Burnside 1005

1 (a) State whether each of the following functions defines an Exponential Family distribution. Where it is possible, write the distribution in the Exponential Family form, and find the natural (canonical) parameterization. If the function does not specify an Exponential Family distribution, explain why not.

(i) The continuous $Uniform(\theta_1, \theta_2)$ distribution:

$$f_X(x|\theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \qquad \theta_1 < x < \theta_2$$

and zero otherwise, for parameters $\theta_1 < \theta_2$.

(ii) The distribution defined by

$$f_X(x|\theta) = \frac{-1}{\log(1-\theta)} \frac{\theta^x}{x} \qquad x = 1, 2, 3, \ldots$$

and zero otherwise, for parameter θ, where $0 < \theta < 1$.

(iii) The distribution defined by

$$f_X(x|\phi, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\phi^2}{2\lambda}x + \phi - \frac{\lambda}{2x}\right\} \qquad x > 0$$

and zero otherwise, for parameters $\phi, \lambda > 0$.

Find the expectation of $1/X$ in terms of ϕ and λ.

8 MARKS

(b) For random variable X, consider a one parameter Exponential Family distribution in its natural parameterization with $k = 1$,

$$f_X(x|\eta) = h(x)c^\star(\eta)\exp\{\eta\, t(x)\}$$

and natural parameter space $\mathcal{H}$. Suppose that $\mathcal{H}$ is an open interval in $\mathbb{R}$, so that for every $\eta \in \mathcal{H}$, there exists an $\epsilon > 0$ such that

$$\eta' \in \mathcal{H} \quad \text{if} \quad |\eta - \eta'| < \epsilon$$

(i) By considering the exponential tilting construction of the Exponential Family, show that the cumulant generating function of random variable $T = t(X)$ under the probability model f_X takes the form

$$K_T(s) = \kappa(\eta + s) - \kappa(s)$$

for $s \in (-h, h)$, some $h > 0$, where κ is some function to be identified.

4 MARKS

(ii) Suppose that $\eta_1, \eta_2 \in \mathcal{H}$. Find the form of the log likelihood ratio, $\ell(x; \eta_1, \eta_2)$, where

$$\ell(x; \eta_1, \eta_2) = \log \frac{f_X(x|\eta_1)}{f_X(x|\eta_2)}.$$

2 MARKS

Turn over for Question 2.

2 Consider the three-level hierarchical model:

LEVEL 3 :	$\mu \in \mathbb{R}, \tau, \sigma > 0$	Fixed parameters
LEVEL 2 :	$M \sim Normal(\mu, \tau^2)$	
LEVEL 1 :	$X_1, X_2 \vert M = m \sim Normal(m, \sigma^2)$	

where X_1 and X_2 are conditionally independent given M, denoted

$$X_1 \perp X_2 \mid M.$$

Using the law of iterated expectation, find the (marginal) covariance and correlation between X_1 and X_2.

5 MARKS

Are X_1 and X_2 (marginally) independent ? Justify your answer.

1 MARK