MATH 556 - MID-TERM EXAMINATION

Marks can be obtained by answering all questions. All questions carry equal marks.

1. (a) Suppose that U is a continuous random variable, and $U \sim Uniform(0,1)$. Let random variable X be defined in terms of U by

$$X = \sin(\pi U/2).$$

Find

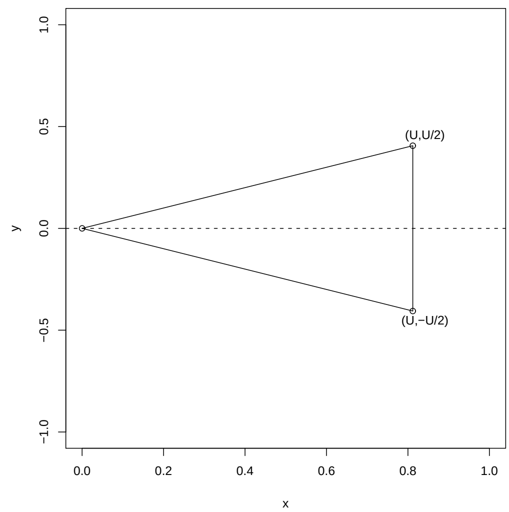
(i) the pdf of X, f_X,

(ii) the expectation $E_{f_X}[X]$,

(iii) the expected area of the (random) triangle $\mathcal{U}$ with corners

$$(0,0), (U, U/2), (U, -U/2)$$

depicted in the following figure.



(b) The joint pdf of continuous random variables Y and Z is specified via the conditional distribution of Y given $Z = z$, and the marginal distribution for Z. Specifically,

$$\begin{aligned} Y|Z = z &\sim Uniform(0, \sqrt{z}) \\ Z &\sim Gamma(3/2, \lambda) \end{aligned}$$

for parameter $\lambda > 0$. Find the marginal pdf for Y, f_Y.

2. Continuous random variables R, X and Y have joint density specified in the following way: the marginal pdf for R, f_R, is defined by

$$f_R(r) = 4r^3 \qquad 0 < r < 1$$

and zero otherwise, and, for $0 < r < 1$, the joint conditional pdf for X and Y **given that** $R = r$, denoted $f_{X,Y|R}$, is given by

$$f_{X,Y|R}(x, y|r) = k(r) \qquad -r < x < r,\ -r < y < r,\ 0 < x^2 + y^2 < r^2.$$

and zero otherwise, where normalizing constant $k(r)$ **depends on** r.

(a) Find the form of $k(r)$, for $0 < r < 1$.

(b) Find the joint marginal pdf for X and Y, denoted $f_{X,Y}$.

(c) By inspecting the form of the joint pdf $f_{X,Y}$, deduce the value of the covariance between X and Y. Are X and Y independent ? Justify your answer.

3. (a) Suppose that Z_1 and Z_2 are independent $Normal(0, 1)$ random variables.

Find the marginal distributions of random variables U and V where

$$U = Z_1 + Z_2 \qquad V = \frac{Z_1}{Z_2}$$

Are U and V independent ? Justify your answer.

(b) Let the marginal pdf f_V from part (a) be the standard member, henceforth denoted f, of the *location-scale* family indexed by parameters (θ, σ).

(i) Write down the form of the pdf of the general member of this location-scale family, $f(x|\theta, \sigma)$, and show that this function is symmetric about θ.

(ii) Write down the expectation derived from the pdf $f(x|\theta, \sigma)$.

4. (a) This question refers to the negative binomial distribution in its "alternative form", where the support of the pmf is $\{0, 1, 2, \ldots\}$.

(i) Write the negative binomial pmf in the form of an *exponential family distribution*, using indicator function notation to identify the support of the pmf.

(ii) Identify the *natural* or *canonical* parameter for the negative binomial distribution.

(iii) Show that the negative binomial distribution is *infinitely divisible.*

(b) The hazard function, h_X, for a continuous random variable X with pdf f_X and cdf F_X is given by

$$h_X(x) = \frac{f_X(x)}{1 - F_X(x)}$$

(i) Find the hazard function for the $Weibull(\alpha, \beta)$ distribution.

(ii) Is the Weibull distribution a member of the exponential family ? Justify your answer.