MATH 556 - EXERCISES 1: SOLUTIONS

1. Need $\displaystyle\sum_{x=1}^{\infty} f_X(x) = 1$. Hence

$$\text{(a)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{2^x} = 1 \qquad \text{(b)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{x2^x} = \log 2$$

$$\text{(c)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6} \qquad \text{(d)} \quad c^{-1} = \sum_{x=1}^{\infty} \frac{2^x}{x!} = e^2 - 1$$

(a) is given by the sum of a geometric progression; (b) uses a logarithmic series; we have

$$\frac{1}{1-t} = 1 + t + t^2 + \ldots = \sum_{x=0}^{\infty} t^x \Longrightarrow -\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \ldots = \sum_{x=1}^{\infty} \frac{t^x}{x}$$

by integrating both sides with respect to t. Hence for $t = 1/2$, we have

$$\log 2 = -\log(1 - 1/2) = \sum_{x=1}^{\infty} \frac{1}{x2^x}.$$

(c) is a well-known mathematical result (see, for example, `http://www.secam.ex.ac.uk/~rjc/rjc.html`, *Evaluating* $\zeta(2)$); (d) uses the power series expansion of e^t, evaluated at $t = 2$, that is

$$e^t = \sum_{x=0}^{\infty} \frac{t^x}{x!} \Longrightarrow e^2 = \sum_{x=0}^{\infty} \frac{2^x}{x!} = 1 + \sum_{x=1}^{\infty} \frac{2^x}{x!}$$

Clearly $\mathrm{P}[\, X > 1 \,] = 1 - \mathrm{P}[\, X = 1 \,]$, so

$$\text{(a)} \quad \mathrm{P}[\, X > 1 \,] = \frac{1}{2} \qquad \text{(b)} \quad \mathrm{P}[\, X > 1 \,] = 1 - \frac{1}{2\log 2}$$

$$\text{(c)} \quad \mathrm{P}[\, X > 1 \,] = 1 - \frac{6}{\pi^2} \qquad \text{(d)} \quad \mathrm{P}[\, X > 1 \,] = \frac{e^2 - 3}{e^2 - 1}$$

$\mathrm{P}[X \text{ is even }] = \displaystyle\sum_{x=1}^{\infty} \mathrm{P}[\, X = 2x \,]$, so

$$\text{(a)} \quad \mathrm{P}[\, X \text{ is even }] = \frac{1}{3} \qquad \text{(b)} \quad \mathrm{P}[\, X \text{ is even }] = 1 - \frac{\log 3}{\log 4}$$

$$\text{(c)} \quad \mathrm{P}[\, X \text{ is even }] = \frac{1}{4} \qquad \text{(d)} \quad \mathrm{P}[\, X \text{ is even }] = \frac{1 - e^{-2}}{2}$$

(a) is still the sum of a geometric progression

(b) follows from the logarithmic series expansion;

$$P[X \text{ is even}] = \sum_{x=1}^{\infty} P[X = 2x] = c\sum_{x=1}^{\infty} \frac{1}{(2x)2^{2x}} = \frac{c}{2}\sum_{x=1}^{\infty} \frac{1}{x4^x} = \frac{c}{2} \times (-\log(1 - 1/4))$$

(c) follows from the initial result taking out a factor of $1/4$; in this case

$$P[X=2x] = \frac{c}{(2x)^2} = \frac{1}{4}\frac{c}{x^2} = \frac{1}{4}P[X=x] \quad \therefore \quad \sum_{x=1}^{\infty} \mathrm{P}[\,X=2x\,] = \frac{1}{4}\sum_{x=1}^{\infty} \mathrm{P}[\,X=x\,] = \frac{1}{4}$$

(d) uses the sum of the two power series of e^t and e^{-t}, to knock out the odd terms, evaluated at $t=2$.

2. Two methods of proof, the first one mechanical, the second using a shortcut. First, let Z and X be the numbers of Heads obtained on the first and second tosses respectively. Then the ranges of Z and X are both $\{0,1,2,...,n\}$. Now

$$f_X(x) = \mathrm{P}[\,X=x\,] = \sum_{z=1}^{n} \mathrm{P}[\,X=x \mid Z=z\,]\,\mathrm{P}[\,Z=z\,] = \sum_{z=x}^{n} \binom{z}{x}\left(\frac{1}{2}\right)^z \binom{n}{z}\left(\frac{1}{2}\right)^n$$

using the Theorem of Total probability. Hence

$$f_X(x) = \left(\frac{1}{2}\right)^n \sum_{z=x}^{n} \frac{z!}{x!(z-x)!}\frac{n!}{z!(n-z)!}\left(\frac{1}{2}\right)^z = \left(\frac{1}{2}\right)^n \binom{n}{x} \sum_{z=x}^{n} \binom{n-x}{n-z}\left(\frac{1}{2}\right)^z$$

But

$$\sum_{z=x}^{n} \binom{n-x}{n-z}\left(\frac{1}{2}\right)^z = \sum_{t=0}^{m} \binom{m}{m-t}\left(\frac{1}{2}\right)^{t+x} = \left(\frac{1}{2}\right)^x \left(1+\frac{1}{2}\right)^m$$

where $t = z-x$, and $m = n-x$, using the Binomial Expansion. Hence

$$f_X(x) = \binom{n}{x}\left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^x \left(1+\frac{1}{2}\right)^{n-x} = \binom{n}{x}\frac{3^{n-x}}{2^{2n}} \qquad x = 0,1,2,...,n.$$

Alternately, as all tosses are independent, consider tossing all n coins twice, and counting the number that show heads twice; this is identical to evaluating X. Then as each coin shows heads twice with probability $(1/2)^2$,

$$f_X(x) = \binom{n}{x}\left\{\left(\frac{1}{2}\right)^2\right\}^x \left\{1-\left(\frac{1}{2}\right)^2\right\}^{n-x} = \binom{n}{x}\frac{3^{n-x}}{2^{2n}}$$

for $x = 0,1,2,...,n$ and zero otherwise, as before.

3. Need to check the properties of a cdf (essentially a nondecreasing function with limiting values 0 and 1 as the argument takes its limiting values at minus or plus infinity). Thus

(a)	Valid cdf	(b)	Valid cdf
(c)	Valid cdf	(d)	Valid cdf

Note in particular that the derivative of each $F(x)$ is positive at all x.

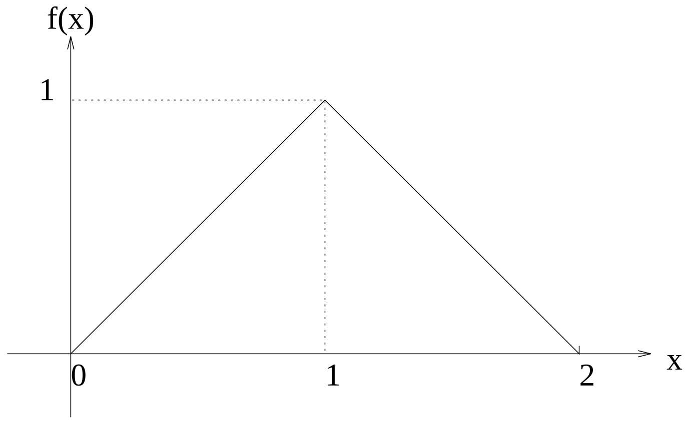
4. Can calculate F_X by integration

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x ct^2(1-t)\,dt = c\left[\frac{x^3}{3} - \frac{x^4}{4}\right] \qquad 0 < x < 1$$

and $F_X(1) = 1$ gives $c = 12$. Finally,

$$\mathrm{P}[\,X > 1/2\,] = 1 - \mathrm{P}[X \le 1/2] = 1 - F_X(1/2) = 1 - 12[1/24 - 1/64] = 11/16.$$

5. Sketch of f_X;



$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} \displaystyle\int_0^x t\,dt & = \dfrac{x^2}{2} & 0 < x < 1 \\[2ex] \displaystyle\int_0^1 t\,dt + \int_1^x (2-t)\,dt & = 2x - \dfrac{x^2}{2} - 1 & 1 \le x < 2 \end{cases}$$

Note that F_X is continuous, and $F_X(0) = 0$, $F_X(2) = 1$.

6. By the usual properties, $F_X(1) = 1 \Longrightarrow c = 1/(\alpha - \beta)$, and

$$f_X(x) = \frac{d}{dt}\{F_X(t)\}_{t=x} = \frac{\alpha\beta}{\alpha - \beta}\left(x^{\beta-1} - x^{\alpha-1}\right) \qquad 0 \le x \le 1$$

and zero otherwise, and hence

$$\mathrm{E}_{f_X}[\,X^r\,] = \int_{-\infty}^{\infty} x^r f_X(x)\,dx = \int_0^1 \frac{\alpha\beta}{\alpha-\beta} x^r \left(x^{\beta-1} - x^{\alpha-1}\right)\,dx = \frac{\alpha\beta}{\alpha-\beta}\left[\frac{x^{\beta+r}}{\beta+r} - \frac{x^{\alpha+r}}{\alpha+r}\right]_0^1 = \frac{\alpha\beta}{(\alpha+r)(\beta+r)}$$

7. By differentiation,

$$f_X(x) = \frac{d}{dt}\{F_X(t)\}_{t=x} = \frac{2\beta(\beta^2 - x^2)}{(\beta^2 + x^2)^2} \qquad 0 \le x \le \beta$$

and zero otherwise, and hence

$$\begin{aligned} \mathrm{E}_{f_X}[\,X\,] &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^\beta x \frac{2\beta(\beta^2 - x^2)}{(\beta^2+x^2)^2}\,dx \\ &= \int_0^{\pi/4} 2\beta^2 \tan\theta \frac{\beta^2(1-\tan^2\theta)}{\beta^4(1+\tan^2\theta)^2} \beta\sec^2\theta\,d\theta \qquad (x = \beta\tan\theta) \\ &= 2\beta\int_0^{\pi/4} \tan\theta\,\frac{(1-\tan^2\theta)}{(1+\tan^2\theta)}\,d\theta = 2\beta\int_0^{\pi/4} \tan\theta\cos 2\theta\,d\theta \\ &= 2\beta\left[\frac{1}{2}\tan\theta\sin 2\theta\right]_0^{\pi/4} - \beta\int_0^{\pi/4}\sec^2\theta\sin 2\theta\,d\theta \qquad \text{(by parts)} \\ &= 2\beta\left[\frac{1}{2} - \int_0^{\pi/4}\tan\theta\,d\theta\right] = 2\beta\left[\frac{1}{2} - [-\log(\cos\theta)]_0^{\pi/4}\right] \\ &= 2\beta\left[\frac{1}{2} + \log(\cos(\pi/4))\right] = \beta(1 - \log 2) \qquad \text{as } \cos(\pi/4) = 1/\sqrt{2}. \end{aligned}$$