MATH 556 - PRACTICE EXAM QUESTIONS II

1. Suppose that $X_1, X_2, ...$ are i.i.d $Cauchy$ random variables with pdf

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad x \in \mathbb{R}$$

and characteristic function $C_X(t) = \exp\{-|t|\}$.

(a) Find the distribution of the random variable T_n defined by

$$T_n = \sum_{i=1}^{n} X_i.$$

(b) Does the sample mean random variable $\overline{X}_n = T_n/n$ converge in probability to zero as $n \to \infty$? Justify your answer.

(c) Show that the Cauchy distribution can be constructed as a *scale mixture* of a normal distribution with a Gamma mixing distribution.

2. (a) Show how an exponential family distribution can be constructed by *tilting* a pdf f_X.

(b) Let f_Y be a pdf for random variable Y, and let $s(Y)$ be a transformed version of Y such that $Var_{f_S}[s(Y)] > 0$. Let the set $\mathcal{N}$ be defined by

$$\mathcal{N} \equiv \left\{\theta \in \mathbb{R} : K_S(\theta) = \log\left[\int e^{s(y)\theta} f_Y(y)\ dy\right] < \infty\right\}$$

(i) Show that $0 \in \mathcal{N}$.

(ii) Using Hölder's Inequality, show that $\mathcal{N}$ is a convex set, that is, if $0 \leq \alpha \leq 1$ and $\theta_1, \theta_2 \in \mathcal{N}$, then

$$\alpha\theta_1 + (1-\alpha)\theta_2 \in \mathcal{N}.$$

(iii) Show that $K_S(\theta)$ is a convex function on $\mathcal{N}$, that is, if $0 \leq \alpha \leq 1$ and $\theta_1, \theta_2 \in \mathcal{N}$, then

$$K_S(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha K_S(\theta_1) + (1-\alpha)K_S(\theta_2)$$